

Linear correlations of the divisor function

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Abstract

Motivated by arithmetic applications on the number of points in a bihomogeneous variety and on moments of Dirichlet L -functions, we provide analytic continuation for the series $\mathcal{A}_{\mathbf{a}}(s) := \sum_{n_1, \dots, n_k \geq 1} \frac{d(n_1) \cdots d(n_k)}{(n_1 \cdots n_k)^s}$ with the sum restricted to solutions of a non-trivial linear equation $a_1 n_1 + \cdots + a_k n_k = 0$. The series $\mathcal{A}_{\mathbf{a}}(s)$ converges absolutely for $\Re(s) > 1 - \frac{1}{k}$ and we show it can be meromorphically continued to $\Re(s) > 1 - \frac{2}{k+1}$ with poles at $s = 1 - \frac{1}{k-j}$ only, for $1 \leq j < (k-1)/2$.

As an application, we obtain an asymptotic formula with power saving error term for the number of points in the variety $a_1 x_1 y_1 + \cdots + a_k x_k y_k = 0$ in $\mathbb{P}^{k-1}(\mathbb{Q}) \times \mathbb{P}^{k-1}(\mathbb{Q})$.

1 Introduction

Motivated by some applications which we shall describe below, we consider the Dirichlet series $\mathcal{A}_{\mathbf{a}}(s)$ obtained by adding a linear constraints among the variables of summation when expanding $(\zeta(s)^2)^k$ into a product of k Dirichlet series. More precisely, for $k \in \mathbb{N}_{\geq 2}$, $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}_{\neq 0}^k$, and $\Re(s) > 1 - \frac{1}{k}$ we define $\mathcal{A}_{\mathbf{a}}(s)$ to be

$$\mathcal{A}_{\mathbf{a}}(s) := \sum_{\substack{n_1, \dots, n_k \geq 1, \\ a_1 n_1 + \cdots + a_k n_k = 0}} \frac{d(n_1) \cdots d(n_k)}{(n_1 \cdots n_k)^s} = \sum_{n \geq 1} \frac{h_{\mathbf{a}}(n)}{n^s}, \quad (1.1)$$

where $d(n)$ is the number of divisors of n and where $h_{\mathbf{a}}(n)$ is defined implicitly by the second identity. Notice we can assume that a_1, \dots, a_k don't all have the same sign, since otherwise $\mathcal{A}_{\mathbf{a}}(s) = 0$. The function $\mathcal{A}_{\mathbf{a}}(s)$ can be regarded as a degree 2 analogue of

$$\mathcal{S}_{\mathbf{a}}(s) := \sum_{\substack{n_1, \dots, n_k \geq 1, \\ a_1 n_1 + \cdots + a_k n_k = 0}} \frac{1}{(n_1 \cdots n_k)^s}.$$

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This function is a particular case of the Shintani zeta-function, which was investigated in a series of works by Shintani (see, e.g. [Shi76, Shi78]). In particular, he showed that $\mathcal{S}_{\mathbf{a}}(s)$ admits a meromorphic continuation to \mathbb{C} and studied its special values displaying a connection with the values at $s = 1$ of Hecke L -function of totally real fields.

The value at $s = 1$ of the function $\mathcal{A}_{\mathbf{a}}(s)$ also has an arithmetic interpretation. Indeed, in [Bet15] it was shown that $\mathcal{A}_{\mathbf{a}}(1)$ appears as the leading constant for the moments of a ‘‘cotangent sum’’ related to the Nyman-Beurling criterion for the Riemann hypothesis. More specifically, it was shown that as $q \rightarrow \infty$

$$\sum_{\substack{h=1, \\ (h,q)=1}}^{q-1} c_0(h/q)^{2k} \sim \frac{q^{2k}}{\pi^{4k}} \sum_{\epsilon \in \{\pm 1\}^k} (-1)^{k - \#\{1 \leq i \leq 2k | \epsilon_i = 1\}} \mathcal{A}_{\epsilon}(1)$$

where $c_0(h/q) := -\sum_{m=1}^q \frac{m}{q} \cot(\frac{\pi mh}{q})$. We defer to [Bag, BC13a, BC13b] for more details on c_0 and on its relation with the Nyman-Beurling criterion. Also, we remark that the asymptotic for the moments of $c_0(h/q)$ was previously computed in [MR16a] with a different expression for the leading constant.

In this paper we are interested in the analytic continuation of $\mathcal{A}_{\mathbf{a}}(s)$. For $k = 2$ it is very easy to analytically continue $\mathcal{A}_{\mathbf{a}}(s)$ to a meromorphic function on \mathbb{C} . Indeed, for $a_1, a_2 \in \mathbb{N}$ one has

$$\mathcal{A}_{\mathbf{a}}(s) = \sum_{\substack{n_1, n_2 \geq 1, \\ a_1 n_1 = a_2 n_2}} \frac{d(n_1)d(n_2)}{(n_1 n_2)^s} = \sum_{n \geq 1} \frac{d(a_1 n)d(a_2 n)}{(abn^2)^s} = \eta_{a_1, a_2}(s) \sum_{n \geq 1} \frac{d(n)^2}{n^{2s}} = \eta_{a_1, a_2}(s) \frac{\zeta(2s)^4}{\zeta(4s)}$$

by Ramanujan’s identity, where $\eta_{a_1, a_2}(s)$ is a certain arithmetic factor which is meromorphic in \mathbb{C} with poles all located on the line $\Re(s) = 0$. In the case $k \geq 3$ the coefficients $h_{\mathbf{a}}(n)$ in (1.1) are no longer multiplicative and the problem of providing meromorphic continuation for $\mathcal{A}_{\mathbf{a}}(s)$ becomes significantly harder, but we are still able to enlarge the domain of holomorphicity of $\mathcal{A}_{\mathbf{a}}(s)$ to $\Re(s) > 1 - \frac{2}{k+1}$.

Theorem 1. *Let $k \geq 3$, $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}_{\neq 0}^k$ with a_1, \dots, a_k not all with the same sign. Then, $\mathcal{A}_{\mathbf{a}}(s)$ admits meromorphic continuation to $\Re(s) > 1 - \frac{2}{k+1}$. More precisely, there exist $c_{m,j}(\mathbf{a}) \in \mathbb{R}$ for $\frac{k+2}{2} \leq m \leq k$, $1 \leq j \leq m+1$ such that*

$$\mathcal{F}_{\mathbf{a}}(s) := \mathcal{A}_{\mathbf{a}}(s) - \sum_{\frac{k+2}{2} \leq m \leq k} \left(\frac{c_{m,m+1}(\mathbf{a})}{(s - (1 - \frac{1}{m}))^{1+m}} + \dots + \frac{c_{m,1}(\mathbf{a})}{s - (1 - \frac{1}{m})} \right)$$

is holomorphic on $\Re(s) > 1 - \frac{2}{k+1}$. Moreover, there exists an absolute constant $A > 0$ such that for $\Re(s) > 1 - \frac{2-\varepsilon}{k+1}$ one has $\mathcal{F}_{\mathbf{a}}(s) \ll_{\varepsilon} \left((\frac{k}{\varepsilon} \max_{m=1}^k |a_m|)^A (1 + |s|)^7 \right)^{Ak^2(1-1/k-\sigma)+k\varepsilon}$, where the implicit constant depends on ε only.

Notice that Theorem 1 is uniform in k , \mathbf{a} and s . We remark that the uniformity in k of some bounds for $\mathcal{A}_{\mathbf{a}}(s)$ at $s = 1$ was crucial in the works [Bet15] and [MR16b] and, in general, it is also needed for our application [Bet].

The value of the arithmetic factors $c_{m,j}(\mathbf{a})$ can be computed explicitly starting from equation (3.2) below. In particular, for $m = k$, $j = k + 1$ one has

$$c_{k,k+1}(\mathbf{a}) = \frac{\rho(\mathbf{a})}{|a_1 \cdots a_k|^{\frac{1}{k}}} \frac{k!}{k^{k+1}} \frac{\zeta(k-1)}{\zeta(k)} \frac{\Gamma(\frac{1}{k})^k}{\Gamma(\frac{r}{k})\Gamma(1-\frac{r}{k})}, \quad (1.2)$$

where r is the number of a_i which are positive and $\rho(\mathbf{a})$ is as defined in (3.5); in particular if $\text{GCD}(a_1, \dots, a_k) = 1$ then $1 \ll \rho(\mathbf{a}) \ll_{\varepsilon} |a_1 \cdots a_k|^{\varepsilon}$. Also, for $1 \leq j \leq k$ one has that $c_{k,j}(\mathbf{a})$ can be written in terms of an arithmetic factor of shape similar to $\rho(\mathbf{a})$ times an expression depending on Euler's constant γ , the derivatives of $\zeta(s-1)$ and $\zeta(s)$ computed at k , and the derivatives of the Γ -function computed at $\frac{1}{k}$, $\frac{r}{k}$ and $1 - \frac{r}{k}$ (cf. equation (3.3)). To give an explicit numerical example for $k = 3$, $\mathbf{a} = (-1, 1, 1)$, we have

$$c_{3,4} \approx 0.537228, \quad c_{3,3} \approx 0.554669, \quad c_{3,2} \approx 1.689055, \quad c_{3,1} \approx -64.704169.$$

An interesting question left open by Theorem 1 is whether $\mathcal{A}_{\mathbf{a}}(s)$ extends meromorphically further to the left, perhaps to a meromorphic function on \mathbb{C} with poles at $s = 1 - \frac{1}{m}$ for all $1 \leq m \leq k$. As an approach to this problem one could try to input a recursive argument into the proof of Theorem 1. We notice however that the expression for the coefficients $c_{m,j}$ arising in the proof of Theorem 1 does not visibly extend to a meaningful formula in the case $m \leq \frac{1}{2}(\sqrt{4k+1} + 1)$ for $k > 3$ ($m = 1$ if $k = 3$), thus suggesting these coefficients might change form at some point or perhaps casting doubts on the possibility of a meromorphic continuation of $\mathcal{A}_{\mathbf{a}}(s)$ to \mathbb{C} . Finally we mention that numerical computations in the case $k = 3$ suggest there is a pole also at the subsequent expected location $s = \frac{1}{2}$ (i.e. a term of order $P(\log X)X^{\frac{3}{2}}$ in (1.4) below), however the computations do not clarify whether the corresponding coefficients have the same shape of the previous coefficients or not.

Our first application of Theorem 1 is given in the following Corollary.

Corollary 1. *Let $k \geq 3$ and $\mathbf{a} \in \mathbb{Z}_{\neq 0}^k$. Let $\Phi(x)$ be a smooth function with support in $[-1, 1]$ and such that $\phi^{(j)}(x) \ll j^{Bj}$ for some $B > 0$ and all $x \in \mathbb{R}$. Then, for $\frac{k+1}{2} < i \leq k$ there exist polynomials $P_{\mathbf{a},i}(x) \in \mathbb{R}[x]$ of degree i such that for all $X \geq 1$*

$$\sum_{\substack{n_1, \dots, n_k \geq 1, \\ a_1 n_1 + \dots + a_k n_k = 0}} d(n_1) \cdots d(n_k) \Phi\left(\frac{n_1 \cdots n_k}{X^k}\right) = \sum_{\frac{k+2}{2} \leq i \leq k} P_{\mathbf{a},i}(\log X) X^{k-\frac{k}{i}} + O_{\varepsilon, \Phi}\left(\left(\frac{k^B}{\varepsilon} \max_{i=1}^k |a_i|\right)^{Ak} X^{k-2+\frac{2+\varepsilon}{k+1}}\right), \quad (1.3)$$

for some absolute constant $A > 0$.

To give two examples, in the cases $k = 3$ and $k = 4$ (with $a_1 = -1$) Corollary 1 gives

$$\sum_{n_1, n_2 \geq 1} d(n_1)d(n_2)d(an_1 + bn_2)\Phi(n_1n_2(an_1 + bn_2)/X^3) = Q_{3,a,b}(\log X)X^2 + O_{\varepsilon,\Phi}((a+b)^A X^{\frac{3}{2}+\varepsilon}), \quad (1.4)$$

$$\sum_{n_1, n_2, n_3 \geq 1} d(n_1)d(n_2)d(n_3)d(an_1 + bn_2 + cn_3)\Phi(n_1n_2(an_1 + bn_2 + cn_3)/X^3) = R_{4,a,b,c}(\log X)X^3 + R_{3,a,b,c}(\log X)X^{\frac{8}{3}} + O_{\varepsilon,\Phi}((a+b+c)^A X^{\frac{12}{5}+\varepsilon}). \quad (1.5)$$

for any $a, b, c \in \mathbb{N}$ and where $Q_{3,a,b}(x)$, $R_{3,a,b,c}(x)$ and $R_{4,a,b,c}(x)$ are polynomials of degree 3, 3 and 4 respectively and A is an absolute constant.

We remark that one could use an easier argument to give the leading term in the asymptotic for the left hand side of (1.3). In fact the main difficulty of Corollary 1 lies in unravelling the complicated combinatorics required to obtain the full main term $P_{a,i}(\log X)X^{k-1}$. These difficulties are implicitly treated in Theorem 1, which allows us to go even further than the full main term. Indeed, for $k \geq 4$ we are able to identify also some new terms whose order is a power smaller than the main term (cf. (1.5)). This is an example of an arithmetic stratification, where one has other “main terms”, coming from sub-varieties, of order (typically) different from the main term one expects from the variety under consideration. This phenomenon was discussed by Manin and Tschinkel [MT] and explored in the context of the circle method by Vaughan and Wooley in the Appendix of [VW]. Recently, the arithmetic stratification was also indicated by Wooley as a potential source for the various terms in the Conrey-Keating analysis [CK] of the asymptotic for moments of the Riemann zeta-function. In our case, the lower order contribution could be explained as coming from affine sub-varieties, that is solutions of $n_1m_1 + \dots + n_k m_k = 0$ which also satisfy one or more other equations $r_1n_1m_1 + \dots + r_k n_k m_k = r_0$ for some “small” $r_0, \dots, r_k \in \mathbb{Z}$.

A result similar to (1.4), with the significant difference in the different way of counting, was obtained by Browning [Bro]. He computed the asymptotic with power-saving error term for

$$\sum_{n_1, n_2 < X} d(L(n_1, n_2))d(L_2(n_1, n_2))d(L_3(n_1, n_2)), \quad (1.6)$$

where $L_1, L_2, L_3 \in \mathbb{Z}[x_1, x_2]$ are linearly independent linear forms. He also considered (1.6) when the sums are unbalanced, i.e. where the sum is restricted to $n_1 \leq N_1$, $n_2 \leq N_2$ with N_2 smaller than N_1 . He was able to prove the asymptotic as long as $N_1^{3/4+\varepsilon} \leq N_2 \leq N_1$ (for $L_1(n_1, n_2) = n_1 - n_2$, $L_2(n_1, n_2) = n_1$, $L_3(n_1, n_2) = n_1 + n_2$), a range that was recently enlarged by Blomer [Blo] who was able to consider the case $N_1^{1/3+\varepsilon} \leq N_2 \leq N_1$ (with a smooth cut-off for n_2). In a different direction, we also cite the work of Browning and de la Bretèche [dlBB], who considered the case $k = 3$ with a quadratic relation among the variables.

For larger values of k , we cite the important work of Matthiesen [Mat] who considered a variation of (1.3) as well as the more general case when one has more than one linear

constraint. Her work differs from ours in that in her case the variables vary inside a convex set, whereas in our case the variables are essentially summed over the hyperbolic region. Also, her method is based on the Green-Tao transference principle [GT] which can only give the leading term $cX^{k-1}(\log X)^k$ in the asymptotic formula. In particular we notice that neither the work of Matthiesen nor those of Browning and Blomer were able to produce terms of order a lower power of X .

Before introducing our second application, we first mention that we shall actually prove a more general version of Theorem 1 where shifts are introduced, i.e. where instead of each divisor function $d(n)$ we have $\tau_{\alpha_i, \beta_i}(n) := \sum_{ab=n} a^{-\alpha_i} b^{-\beta_i}$ with $\alpha_i, \beta_i \in \mathbb{C}$. We defer to Theorem 3 in Section 2 for the complete statement. The shifts make our result much more flexible. In particular we can use it to count integer solutions $(x_1, \dots, x_k, y_1, \dots, y_k)$ in the flag variety

$$a_1 x_1 y_1 + \dots + a_k x_k y_k = 0 \tag{1.7}$$

when ordered according to the size of the anticanonical height $(\max_i |x_i| \cdot \max_j |y_j|)^{\frac{1}{k-1}}$, verifying Manin's conjecture in this particular case.

Theorem 2. *Let $k \in \mathbb{N}_{\geq 3}$. For $\mathbf{x} \in \mathbb{P}^{k-1}(\mathbb{Q})$, let (x_1, \dots, x_k) be a representative of \mathbf{x} such that $x_1, \dots, x_k \in \mathbb{Z}$ and $(x_1, \dots, x_k) = 1$. Let $H : \mathbb{P}^{k-1}(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$ be defined by $H(\mathbf{x}) := (\max_{1 \leq i \leq k} |x_i|)^{\frac{1}{k-1}}$ and let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}_{\neq 0}^k$. Then,*

$$\begin{aligned} N(B) &:= \#\left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{P}^{k-1}(\mathbb{Q}) \times \mathbb{P}^{k-1}(\mathbb{Q}) \left| \begin{array}{l} a_1 x_1 y_1 + \dots + a_k x_k y_k = 0, H(\mathbf{x})H(\mathbf{y}) < B \\ x_1 \cdots x_k \cdot y_1 \cdots y_k \neq 0 \end{array} \right. \right\} \\ &= \left(\frac{\mathfrak{S}(\mathbf{a})}{k-1} \sum_{i=1}^k \sigma_i(\mathbf{a}) \right) B \log B + f(\mathbf{a})B + O_{k, \varepsilon, \mathbf{a}}(B^{1 - \frac{k-2}{2k(15k-2)} + \varepsilon}), \end{aligned}$$

for some explicitly computable $f(\mathbf{a}) \in \mathbb{R}$, and

$$\begin{aligned} \mathfrak{S}(\mathbf{a}) &:= \frac{1}{2} \sum_{\ell \geq 1} \frac{\prod_{i=1}^k (a_i, \ell)}{\ell^k} \frac{\varphi(\ell)}{\zeta(k)^2}, \\ \sigma_i(\mathbf{a}) &:= \left(\prod_{i=2}^k \int_{-1}^1 \int_{-1}^1 \right) \chi_{(-|a_i|, |a_i|)} \left(\sum_{j=1, j \neq i}^k a_j x_j y_j \right) \prod_{j=1, j \neq i}^k dx_j dy_j, \end{aligned} \tag{1.8}$$

where χ_X is the characteristic function of the set X .

The variety (1.7) has already been considered in several papers. In particular, we mention the works of Robbiani [Rob] (for $k \geq 3$), Spencer [Spe] and Browning [Bro] (for $k = 2$) and, previously, by Franke, Manin and Tschinzel [FMT] and Thunder [Thu] (with height function

$\|x\|^{\frac{1}{k-1}}\|y\|^{\frac{1}{k-1}}$)¹ in the more general setting of Fano varieties. Among all these works, the only one where the full main term is obtained, with an (implicit) bound of the form $O(X^{1-\varepsilon})$ for the error term, is [FMT] (see the Corollary after Theorem 5). Theorem 2 thus gives a more direct proof of their result for this case, as well as providing an explicit power saving error term.

We notice that Theorem 2 appears very similar to Corollary 1, which essentially counts points in (1.7) ordering them according to the size of the product of all the variables, $|x_1 \cdots x_k \cdot y_1 \cdots y_k|$.² The different way of counting however changes the problem significantly and the deduction of Theorem 2 from (the generalization of) Theorem 1 is much subtler. In particular, the computation of the full main term for $N(B)$ requires a careful analysis of some complex integrals resulting from integrating over the shifts in Theorem 3. Notice that also in this case the problem becomes much easier if one only computes the leading term in the asymptotic for $N(B)$.

A third application of Theorem 1 comes from the theory of the moments of L -functions. In [Bet] it is considered the moment

$$M_k(q) := \sum'_{\chi_1, \dots, \chi_{k-1} \pmod{q}} |L(\frac{1}{2}, \chi_1)|^2 \cdots |L(\frac{1}{2}, \chi_{k-1})|^2 |L(\frac{1}{2}, \chi_1 \cdots \chi_{k-1})|^2,$$

where \sum' indicates that the sum is over primitive characters $\chi_1, \dots, \chi_{k-1}$ modulo q and $L(s, \chi)$ is the Dirichlet L -function associated to the character χ . It turns out that the “diagonal term” in $M_k(q)$ has the shape $\sum_{\epsilon \in \{\pm 1\}^k} c_\epsilon \int_{(2)} \mathcal{A}_\epsilon(s) q^{ks} H(s) ds$ for a meromorphic function $H(s)$ and some $c_\epsilon \in \mathbb{R}$. Thanks to Theorem 1 we are able to evaluate the diagonal term and thus, evaluating also the off-diagonal term using a similar method, we are able to obtain the following asymptotic formula for $M_k(q)$ when $k \geq 3$ (the case $k = 2$ corresponds to the 4-th moment of Dirichlet L -function and was computed by Young [You])

$$M_k(q) = \varphi(q)^{k-1} \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^{\frac{k}{2}}} \left((\log \frac{q}{8n\pi})^k + (-\pi)^k \right) + O_\varepsilon(k^{Ak} q^{-\delta_k + \varepsilon}), \quad (1.9)$$

where $\varphi(n)$ is Euler’s ϕ function, $\nu(n)$ is the number of different prime factors of n and $\delta_k > 0$. We also mention that, thanks to the work [Bet16], (1.9) can be interpreted also as the moment of some functions involving continued fractions.

The proof of Theorem 1 is quite simple in spirit but it has to face a number of technical challenges, mainly coming from the identification of the polar structure (equivalently, of the main terms in (1.3)). Before giving a brief outline of our proof, we mention that one could have chosen to proceed also in different ways, for example using the circle method. The main

¹Here $\|\cdot\|$ is the Euclidean norm and $\mathbf{x} = (x_1, \dots, x_k)$, $\mathbf{y} = (y_1, \dots, y_k)$

²One could easily modify our argument to count points with respect to $\max_i |x_i y_i|$, since our proof starts by introducing partitions of unity which localize each $x_i y_i$.

difficulty however comes from the evaluation of the polar structure and this is not visibly simplified by choosing such different routes. We also remark that the our technique would allow to give analytic continuation also when the constraint is a non-homogeneous linear equation. The only difference with our case is that in Lemma 9 below we would need to use the Deshouillers and Iwaniec [DI] bound for sums of Kloosterman sums (cf. [Bet] where this is done for a similar problem). However, for simplicity we content ourself with dealing with the homogenous case only.

We conclude with a rough sketch of the proof of Theorem 1 referring for simplicity to Corollary 1 which is essentially equivalent to it. First, we split the sum on the right hand side of (1.3) introducing partitions of unity to control the size of the n_i . When one variable is much larger than the others a simple bound suffices, so we are left with considering the case when the variables have about the same size. In this case we eliminate the larger variable using the linear equation and we separate the remaining variables arithmetically and analytically using, respectively, a slightly modified version of Ramanujan's formula,

$$\sigma_{1-s}(m) = \zeta(s) \sum_{\ell \geq 1} \frac{c_\ell(m)}{\ell^s} \quad \Re(s) > 1, \quad (1.10)$$

where $c_\ell(m)$ is the Ramanujan sum and $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$ and a generalized version of the Mellin formula for $(1 \pm x)^{-s}$ as given in [Bet]. We end up with a formula of the shape (for $\mathbf{a} = (1, \dots, 1)$)

$$\sum_{\ell \geq 1} \sum_{\substack{h \pmod{\ell} \\ (h, \ell) = 1}} \sum_{n_1, \dots, n_{k-1} \geq 1} \sum_{n_1, \dots, n_{k-1}} d(n_1) \cdots d(n_{k-1}) e\left(\frac{hn_1 + \dots + hn_{k-1}}{\ell}\right) f(n_1, \dots, n_{k-1}, \ell)$$

for some smooth function f . Applying Voronoi's summation formula to each variable n_1, \dots, n_k transform each sum over n_i in a main term M_i plus a sum of similar shape but with h replaced by \bar{h} and thus we obtain

$$\sum_{\ell \geq 1} \sum_{\substack{h \pmod{\ell} \\ (h, \ell) = 1}} \prod_{i=1}^{k-1} \left(M_i + \sum_{n_i \geq 1} d(n_i) e\left(\frac{\bar{h}n_i}{\ell}\right) f_i(\dots) \right)$$

for some smooth functions f_i . We then treat as main terms the terms where we pick up more M_i than series, and treat the other terms as error terms which we estimate essentially trivially. We then treat and assembly the main terms (which correspond to the poles of $\mathcal{A}_{\mathbf{a}}(s)$), an operation which constitutes the main difficulty of the paper as we have to deal with several integral transforms in order to take them to their final form (actually, we choose the equivalent root of moving the lines of integration of several complex integrals, collecting the contribution of the residues of some poles). Combining the two cases for the range of the variables one then deduces Corollary 1.

We notice that the above structure of the proof of Theorem 1 is at first glance very similar to that of the asymptotic for $M_k(q)$ performed in [Bet]. There are however several important differences at a more detailed level, e.g. in the ways the integrals are manipulated, in the treatment of the error terms and in the combinatorics.

The paper is organized as follow. In Section 2 we state Theorem 3 which gives the analytic continuation for the shifted version of $\mathcal{A}_a(s)$ and in Section 3 we easily deduce Theorem 1 from it and we compute the constants given in (1.2). In Section 4 we prove Theorem 2 by integrating over the shifts introduced in Theorem 3 and evaluating the resulting complex integrals. The rest of the paper is dedicated to the proof of Theorem 3: in Section 5 we give a uniform bound for the region of absolute convergence, whereas in Section 6 we set up the proof of Theorem 3 dividing the sum according to the range of the variables. In Section 7 we estimate the case where the variables have roughly the same size and in Section 8 we give a trivial bound for the case where there's a large variable. Finally, in Section 9 we recombine the various sums reconstructing the polar terms.

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2 The shifted case

For $k \geq 3$, \mathbf{a} as above and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \in \mathbb{C}^k$ we define the Dirichlet series

$$\mathcal{A}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s) := \sum_{a_1 n_1 + \dots + a_k n_k = 0} \frac{\tau_{\alpha_1, \beta_1}(n_1) \cdots \tau_{\alpha_k, \beta_k}(n_k)}{n_1^{\frac{1}{2}+s} \cdots n_k^{\frac{1}{2}+s}},$$

where $\tau_{\alpha, \beta}(n) := \sum_{d_1 d_2 = n} d_1^{-\alpha} d_2^{-\beta}$. If $|\Re(\alpha_m)|, |\Re(\beta_m)| \leq \frac{1}{2(k-1)}$ for all m , then it is easy to see (cf. Lemma 4 below) that $\mathcal{A}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ converges absolutely on

$$\Re(s) > 1 - \frac{1}{k} - \frac{1}{k} \sum_{m=1}^k \min(\Re(\alpha_m), \Re(\beta_m)).$$

The following Theorem gives the analytic continuation for $\mathcal{A}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ to a larger half-plane, provided that

$$\eta_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \frac{3}{2} \sum_{m=1}^k (|\Re(\alpha_m)| + |\Re(\beta_m)|) \tag{2.1}$$

is not too large. Before stating the theorem we need to introduce (a slight variation of) the Estermann function, which for $\alpha, \beta \in \mathbb{C}$, $h \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ is defined as

$$D_{\alpha, \beta}(s, \frac{h}{\ell}) := \sum_{n \geq 1} \frac{\tau_{\alpha, \beta}(n)}{n^s} e(n \frac{h}{\ell})$$

for $\Re(s) > 1 - \min(\Re(\alpha), \Re(\beta))$ and where $e(x) := e^{2\pi i x}$. The Estermann function can be continued to a meromorphic function on \mathbb{C} satisfying a functional equation (see e.g. [BC13b]).

Lemma 2. *Let $\alpha, \beta \in \mathbb{C}$, $h \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ with $(h, \ell) = 1$. Then*

$$D_{\alpha, \beta}(s, \frac{h}{\ell}) - \ell^{1-\alpha-\beta-2s} \zeta(s + \alpha) \zeta(s + \beta)$$

can be extended to an entire function of s . Moreover, one has

$$D_{\alpha, \beta}(s, \frac{h}{\ell}) = \ell^{1-2s-\alpha-\beta} (\chi_{+1}(s; \alpha, \beta) D_{-\alpha, -\beta}(s, \frac{\bar{h}}{\ell}) - \chi_{-1}(s; \alpha, \beta) D_{-\alpha, -\beta}(s, -\frac{\bar{h}}{\ell})), \quad (2.2)$$

where \bar{h} denotes the inverse of $h \pmod{\ell}$ and

$$\chi_{\pm 1}(s; \alpha, \beta) := 2(2\pi)^{2s-2+\alpha+\beta} \Gamma(1-s-\alpha) \Gamma(1-s-\beta) \cos\left(\pi \frac{(s+\alpha) \mp (s+\beta)}{2}\right).$$

For $\alpha, \beta \in \{z \in \mathbb{C} \mid |\Re(z)| < \frac{1}{2(k-1)}\}^k$ we also define

$$\begin{aligned} \mathcal{M}_{\mathbf{a}; \alpha, \beta}(s) := & \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, |\mathcal{I}| \geq \frac{k+2}{2}}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i, \beta_i\} \\ \forall i \in \mathcal{I}}} \left(\prod_{i \in \mathcal{I}} \frac{\zeta(1 - \alpha_i^* + \beta_i^*)}{|a_i|^{-\alpha_i^* + \frac{1+s_{\mathcal{I}, \alpha^*}}{|\mathcal{I}|}}} \Gamma(-\alpha_i^* + \frac{1+s_{\mathcal{I}, \alpha^*}}{|\mathcal{I}|}) \right) \\ & \times \frac{\Delta_{\alpha^*; \mathcal{I}}}{|\mathcal{I}|(s-1) + s_{\mathcal{I}, \alpha^*} + 1} \sum_{\ell \geq 1} \frac{\prod_{i \in \mathcal{I}} (a_i, \ell)^{1-\alpha_i^* + \beta_i^*}}{\ell^{\sum_{i \in \mathcal{I}} (1-\alpha_i^* + \beta_i^*)}} \sum_{h \pmod{\ell}}^* \prod_{j \in \mathcal{J}} D_{\alpha_j, \beta_j}(1 - \frac{1+s_{\mathcal{I}, \alpha^*}}{|\mathcal{I}|}, \frac{h a_j}{\ell}), \end{aligned} \quad (2.3)$$

where \sum^* indicates the the sum is over h which are coprime to ℓ , and the second sum is over $\alpha^* = (\alpha_i^*)_{i \in \mathcal{I}}$, $\beta^* = (\beta_i^*)_{i \in \mathcal{I}} \in \mathbb{C}^{|\mathcal{I}|}$ satisfying the above condition. Also, we put $s_{\mathcal{I}, \alpha^*} := \sum_{r \in \mathcal{I}} \alpha_r^*$ and

$$\Delta_{\alpha^*; \mathcal{I}} := \left(\Gamma\left(\sum_{i \in \mathcal{I}, \text{sign}(a_i)=1} \left(-\alpha_i^* + \frac{1+s_{\mathcal{I}, \alpha^*}}{|\mathcal{I}|} \right) \right) \Gamma\left(\sum_{i \in \mathcal{I}, \text{sign}(a_i)=-1} \left(-\alpha_i^* + \frac{1+s_{\mathcal{I}, \alpha^*}}{|\mathcal{I}|} \right) \right) \right)^{-1}$$

if neither of the two sums inside the Γ functions are empty sums and $\Delta_{\alpha^*; \mathcal{I}} := 0$ otherwise.

Remark 1. Equation (2.3) should be interpreted as defining $\mathcal{M}_{\mathbf{a};\alpha,\beta}(s)$ as a meromorphic function. Also, the definition of $\mathcal{M}_{\mathbf{a};\alpha,\beta}(s)$ can be extended to include the case where $\alpha_i = \beta_i$ since the limit for $\alpha_i \rightarrow \beta_i$ exists (cf. the proof of Theorem 1).

The absolute convergence of the series over ℓ in (2.3) for $\alpha, \beta \in \{z \in \mathbb{C} \mid |\Re(z)| < \frac{1}{2(k-1)}\}^k$ is ensured by the convexity bound for the Estermann function,

$$D_{\alpha,\beta}(s, \frac{h}{\ell}) \ll \delta^{-2} (\ell(1+|s|+|\alpha|))^{\frac{1}{2}} (1+|s|+|\beta|)^{\frac{1}{2}} \ell^{1-\Re(s)-\min(\Re(\alpha), \Re(\beta))+\delta}, \quad (2.4)$$

valid for $\delta > 0$, $\Re(\alpha), \Re(\beta) \ll 1$, $|1-s-\alpha|, |1-s-\beta| > \delta$, and

$$-\max(\Re(\alpha), \Re(\beta)) - \delta \leq \Re(s) \leq 1 - \min(\Re(\alpha), \Re(\beta)) + \delta. \quad (2.5)$$

Indeed, using (2.4) one has that the series over ℓ converges as long as $|\mathcal{I}| > 2 + \frac{|\mathcal{J}|}{|\mathcal{I}|} + \sum_{i \in \mathcal{I}} (\frac{k}{|\mathcal{I}|} \Re(\alpha_i^*) - \Re(\beta_i^*)) - \sum_{j \in \mathcal{J}} \min(\Re(\alpha_j), \Re(\beta_j))$. The right hand side is less than 4, so the only problematic case is when $|\mathcal{I}| = 3$. This can happen only for $k = 3$ and $k = 4$, and in the first case the convergence is clear since there is no Estermann function. Finally, the series converges also for $k = 4$, $|\mathcal{I}| = 3$ and $|\mathcal{J}| = 1$ since one can save an extra factor of $\ell^{1-\varepsilon}$ using the convexity bound for $\sum_h^* D_{\alpha,\beta}(s, \frac{h}{\ell})$

$$\sum_{h \pmod{\ell}}^* D_{\alpha,\beta}(s, \frac{h}{\ell}) \ll_{s,\alpha,\beta,\delta,\varepsilon} \ell^{1-\Re(s)-\min(\Re(\alpha), \Re(\beta))+\delta+\varepsilon},$$

for s satisfying (2.5) and where we used the bound $c_\ell(m) \ll d(\ell)(m, \ell)$ for the Ramanujan sum $c_\ell(m) := \sum_{\ell|a} e(\frac{hm}{\ell})$.

We are now ready to state our main theorem.

Theorem 3. Let $k \geq 3$, $\mathbf{a} \in \mathbb{Z}_{\neq 0}^k$. Then, $\mathcal{A}_{\mathbf{a};\alpha,\beta}(s)$ admits meromorphic continuation to $\Re(s) > 1 - \frac{2-2\eta_{\alpha,\beta}}{k+1}$, $\alpha, \beta \in \{s \in \mathbb{C} \mid |\Re(s)| < \frac{1}{2(k-1)}\}^k$. Moreover, if for some $\varepsilon > 0$ one has $\alpha, \beta \in \{s \in \mathbb{C} \mid |\Re(s)| < \frac{1-\varepsilon}{2(k-1)}\}^k$ and $1 - \frac{2-2\eta_{\alpha,\beta}-\varepsilon}{k+1} \leq \Re(s) \leq 1 - \frac{1-\varepsilon}{k} + \frac{\eta_{\alpha,\beta}}{k+1}$, then

$$\begin{aligned} & \mathcal{A}_{\mathbf{a};\alpha,\beta}(s) - \mathcal{M}_{\mathbf{a};\alpha,\beta}(s) \\ & \ll_\varepsilon \left(\left(\frac{k}{\varepsilon} \max_{i=1}^k |a_i| \right)^A \left((1+|s|) \left(1 + \max_{i=1}^k (|\Im(\alpha_i)| + |\Im(\beta_i)|) \right) \right)^7 \right)^{Ak \frac{(k+1)(1-\frac{1}{k}-\sigma)+\eta_{\alpha,\beta}+\varepsilon}{1-\frac{1}{k}-\eta_{\alpha,\beta}}} \end{aligned} \quad (2.6)$$

for some absolute constant $A > 0$.

Remark 2. Clearly Theorem 3 provides analytic continuation also for

$$\mathcal{A}_{\mathbf{a};\alpha,\beta}^*(s) := \sum_{\substack{n_1, \dots, n_k \in \mathbb{Z}_{\neq 0}, \\ a_1 n_1 + \dots + a_k n_k = 0}} \frac{\tau_{\alpha_1, \beta_1}(|n_1|) \cdots \tau_{\alpha_k, \beta_k}(|n_k|)}{|n_1|^{\frac{1}{2}+s} \cdots |n_k|^{\frac{1}{2}+s}}, \quad (2.7)$$

since $\mathcal{A}_{\mathbf{a};\alpha,\beta}^*(s) = \sum_{\epsilon \in \{\pm 1\}^k} \mathcal{A}_{\mathbf{a}_\epsilon;\alpha,\beta}(s)$, where $\mathbf{a}_\epsilon = (\pm_1 a_1, \dots, \pm_k a)$. Moreover, the sum over ϵ in the ‘‘polar term’’ $\mathcal{M}_{\mathbf{a};\alpha,\beta}^*(s) := \sum_{\epsilon \in \{\pm 1\}^k} \mathcal{M}_{\mathbf{a}_\epsilon;\alpha,\beta}^*(s)$ for $\mathcal{A}_{\mathbf{a};\alpha,\beta}^*(s)$ can be executed giving a neater expression for $\mathcal{M}_{\mathbf{a};\alpha,\beta}^*(s)$. Indeed, for $\sum_{i \in \mathcal{I}} z_i = 1$ we have

$$\begin{aligned}
\sum_{\mathcal{H} \subseteq \mathcal{I}} \frac{\prod_{i \in \mathcal{I}} \Gamma(z_i)}{\Gamma(\sum_{i \in \mathcal{H}} z_i) \Gamma(\sum_{i \in \mathcal{I} \setminus \mathcal{H}} z_i)} &= \left(\prod_{i \in \mathcal{I}} \Gamma(z_i) \right) \sum_{\mathcal{H} \subseteq \mathcal{I}} \frac{\sin(\pi \sum_{i \in \mathcal{H}} z_i)}{\pi} \\
&= \frac{1}{2\pi i} \left(\prod_{i \in \mathcal{I}} \Gamma(z_i) (1 + e^{\pi i z_i}) - \prod_{i \in \mathcal{I}} \Gamma(z_i) (1 + e^{-\pi i z_i}) \right) \\
&= \frac{2^{|\mathcal{I}|}}{2\pi i} \left(e^{\frac{\pi i}{2} \sum_{i \in \mathcal{I}} z_i} \prod_{i \in \mathcal{I}} \Gamma(z_i) \cos\left(\frac{\pi}{2} z_i\right) - e^{-\frac{\pi i}{2} \sum_{i \in \mathcal{I}} z_i} \prod_{i \in \mathcal{I}} \Gamma(z_i) \cos\left(\frac{\pi}{2} z_i\right) \right) \\
&= 2\pi^{\frac{|\mathcal{I}|-1}{2}} \prod_{i \in \mathcal{I}} \frac{\Gamma\left(\frac{z_i}{2}\right)}{\Gamma\left(\frac{1-z_i}{2}\right)}
\end{aligned} \tag{2.8}$$

by the identities for the Γ function $(\Gamma(z)\Gamma(1-z))^{-1} = \frac{1}{\pi} \sin(\pi z)$ and $\cos\left(\frac{\pi z}{2}\right)\Gamma(z) = \frac{\pi^{1/2} 2^{z-1} \Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)}$.

Thus, since $\sum_{i \in \mathcal{I}} (-\alpha_i^* + \frac{1+s_{\mathcal{I},\alpha^*}}{|\mathcal{I}|}) = 1$, it follows that

$$\begin{aligned}
\mathcal{M}_{\mathbf{a};\alpha,\beta}^*(s) &= \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, |\mathcal{I}| \geq \frac{k+2}{2}}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i, \beta_i\} \\ \forall i \in \mathcal{I}}} \left(\prod_{i \in \mathcal{I}} \frac{\zeta(1 - \alpha_i^* + \beta_i^*) \Gamma\left(-\frac{\alpha_i^*}{2} + \frac{1+s_{\mathcal{I},\alpha^*}}{2|\mathcal{I}|}\right)}{|a_i|^{-\alpha_i^* + \frac{1+s_{\mathcal{I},\alpha^*}}{|\mathcal{I}|}} \Gamma\left(\frac{1+\alpha_i^*}{2} - \frac{1+s_{\mathcal{I},\alpha^*}}{2|\mathcal{I}|}\right)} \right) \\
&\times \frac{2^{|\mathcal{J}|+1} \pi^{\frac{|\mathcal{I}|-1}{2}}}{|\mathcal{I}|(s-1) + s_{\mathcal{I},\alpha^*} + 1} \sum_{\ell \geq 1} \frac{\prod_{i \in \mathcal{I}} (a_i, \ell)^{1-\alpha_i^* + \beta_i^*}}{\ell^{\sum_{i \in \mathcal{I}} (1-\alpha_i^* + \beta_i^*)}} \sum_{h \pmod{\ell}}^* \prod_{j \in \mathcal{J}} D_{\cos; \alpha_j, \beta_j} \left(1 - \frac{1+s_{\mathcal{I},\alpha^*}}{|\mathcal{I}|}, \frac{ha_j}{\ell}\right),
\end{aligned} \tag{2.9}$$

where $D_{\cos; \alpha, \beta}(s, \frac{h}{\ell}) := \frac{1}{2} (D_{\alpha, \beta}(s, \frac{h}{\ell}) + D_{\alpha, \beta}(s, -\frac{h}{\ell}))$.

3 The proof of Theorem 1

Remark 3. Throughout the rest of the paper, by a bold symbol \mathbf{v} we indicate the vector $(v_1, \dots, v_k) \in \mathbb{C}^k$. Also, for any set $I \subseteq \{1, \dots, k\}$ by \mathbf{v}_I we indicate the vector $(v_i)_{i \in I} \in \mathbb{C}^{|I|}$.

For any $c \in \mathbb{R}$, by $\int_{(c)} \cdot ds$ we indicate that the integral is taken along the vertical line from $c - i\infty$ to $c + i\infty$. Also, we will often abbreviate $\int_{(c_1)} \cdots \int_{(c_r)}$ with $\int_{(c_1, \dots, c_r)}$.

Finally, by ε we indicate a sufficiently small positive real number, and by A a positive absolute constant whose value might be different at each occurrence.

We now show how to derive Theorem 1 from Theorem 3.

Proof of Theorem 1. First, we assume $|\alpha_i|, |\beta_i| < \frac{\varepsilon}{2k}$ and $\Re(s) > 1$. Then, by the residue theorem

$$\begin{aligned} \mathcal{M}_{\mathbf{a}; \boldsymbol{\alpha}, \boldsymbol{\beta}}(s) &= \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, \\ |\mathcal{I}| \geq \frac{k+2}{2}}} \prod_{i \in \mathcal{I}} \left(\frac{1}{2\pi i} \int_{|w_i| = \frac{\varepsilon}{k}} \frac{\zeta(1 + \alpha_i + w_i) \zeta(1 + \beta_i + w_i)}{|a_i|^{w_i + \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}}} \Gamma(w_i + \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}) \right) \\ &\times \frac{\Delta_{\mathbf{w}_{\mathcal{I}; \mathcal{I}}}}{|\mathcal{I}|(s-1) + s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}} + 1} \sum_{\ell \geq 1} \frac{\prod_{i \in \mathcal{I}} (a_i, \ell)^{1+2w_i + \alpha_i + \beta_i}}{\ell^{\sum_{i \in \mathcal{I}} (1+2w_i + \alpha_i + \beta_i)}} \sum_{h \pmod{\ell}}^* \prod_{j \in \mathcal{J}} D_{\alpha_j, \beta_j} \left(1 - \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}, \frac{ha_j}{\ell} \right) \prod_{i \in \mathcal{I}} dw_i, \end{aligned}$$

where $\mathbf{w}'_{\mathcal{I}} := (-w'_i)_{i \in \mathcal{I}}$, $s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}} = -\sum_{i \in \mathcal{I}} w_i$, and the circles are oriented in the positive direction. Thus, letting $\boldsymbol{\alpha}, \boldsymbol{\beta} \rightarrow \mathbf{0}$ we obtain

$$\begin{aligned} \mathcal{M}_{\mathbf{a}; \mathbf{0}, \mathbf{0}}(s) &= \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, \\ |\mathcal{I}| \geq \frac{k+2}{2}}} \left(\prod_{i \in \mathcal{I}} \frac{1}{2\pi i} \int_{|w_i| = \frac{\varepsilon}{k}} \frac{\zeta(1 + w_i)^2 (a_i, \ell)^{2w_i}}{|a_i|^{w_i + \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}}} \Gamma(w_i + \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}) \right) \\ &\times \sum_{\ell \geq 1} \frac{\prod_{i \in \mathcal{I}} (a_i, \ell)}{\ell^{|\mathcal{I}| + \sum_{i \in \mathcal{I}} 2w_i}} \sum_{h \pmod{\ell}}^* \left(\prod_{j \in \mathcal{J}} D_{0,0} \left(1 - \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}, \frac{ha_j}{\ell} \right) \right) \frac{\Delta_{\mathbf{w}_{\mathcal{I}; \mathcal{I}}}}{|\mathcal{I}|(s-1) + s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}} + 1} \prod_{i \in \mathcal{I}} dw_i. \end{aligned} \quad (3.1)$$

Now, for $w \neq 0$ we have

$$\frac{1}{w - w_1 - \dots - w_k} = \sum_{m=0}^k \frac{m!}{w^{m+1}} \sum_{\substack{S \subseteq \{1, \dots, k\}, \\ |S|=m}} \prod_{i \in S} w_i + O_w \left(\sum_{i=1}^k |w_i|^2 \right)$$

as $w_1, \dots, w_k \rightarrow 0$ and so

$$\frac{1}{|\mathcal{I}|(s-1) + s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}} + 1} = \sum_{m=0}^{|\mathcal{I}|} \frac{m!}{(|\mathcal{I}|(s-1) + 1)^{m+1}} \sum_{S \subseteq \mathcal{I}, |S|=m} \prod_{i \in S} w_i + O_s \left(\sum_{i=1}^m |w_i|^2 \right).$$

Also, if $|w_i| < \frac{\varepsilon}{k}$ for all $i \in \mathcal{I}$ with ε small enough, then we have

$$\begin{aligned} \left(\prod_{i \in \mathcal{I}} \Gamma(w_i + \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}) \right) \Delta_{\mathbf{w}_{\mathcal{I}; \mathcal{I}}} \sum_{h \pmod{\ell}}^* \prod_{j \in \mathcal{J}} D_{0,0} \left(1 - \frac{1+s_{\mathcal{I}, \mathbf{w}'_{\mathcal{I}}}}{|\mathcal{I}|}, \frac{ha_j}{\ell} \right) \\ = \sum_{S \subseteq \mathcal{I}} f_{\mathcal{I}, S}(\ell) \prod_{i \in S} w_i + O_s \left(\sum_{i=1}^m |w_i|^2 \right) \end{aligned}$$

for some $f_{\mathcal{I},S} \in \mathbb{C}$. It follows that

$$\begin{aligned} \mathcal{M}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s) &= \sum_{m=0}^{|\mathcal{I}|} \sum_{\substack{S_1, S_2 \subseteq \mathcal{I}, \\ |S_1|=m, S_1 \cap S_2 = \emptyset}} \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, |\mathcal{I}| \geq \frac{k+2}{2}}} \sum_{\ell \geq 1} \frac{\prod_{i \in \mathcal{I}}(a_i, \ell)}{\ell^{|\mathcal{I}|}} \frac{m! f_{\mathcal{I}, S_2}(\ell)}{(|\mathcal{I}|(s-1) + 1)^{m+1}} \\ &\times \prod_{i \in \mathcal{I}} \operatorname{Res}_{w_i=0} \left(\frac{\zeta(1+w_i)^2 w_i^{\delta_{i \in S_1 \cup S_2}}}{|a_i|^{w_i + \frac{1}{|\mathcal{I}|}} (\ell/(a_i, \ell))^{2w_i}} \prod_{r \in \mathcal{I}} |a_r|^{\frac{w_i}{|\mathcal{I}|}} \right), \end{aligned} \quad (3.2)$$

where $\delta_{i \in S_1 \cup S_2} = 1$ if $i \in S_1 \cup S_2$ and $\delta_{i \in S_1 \cup S_2} = 0$ otherwise. By analytic continuation this gives an expression for $\mathcal{M}_{\mathbf{a};\mathbf{0},\mathbf{0}}(s)$ for all $s \in \mathbb{C}$ and so Theorem 1 follows by Theorem 3. \square

We conclude the section by computing the value of the constant $c_{k,k+1}(\mathbf{a})$.

First we observe that the terms in this sum with $\mathcal{I} = \{1, \dots, k\}$ are

$$\begin{aligned} \sum_{m=0}^k \frac{m!}{(k(s-1) + 1)^{m+1}} \sum_{\substack{S_1, S_2 \subseteq \{1, \dots, k\}, \\ |S_1|=m, S_1 \cap S_2 = \emptyset}} \sum_{\ell \geq 1} \frac{\prod_{i=1}^k (a_i, \ell)}{\ell^k} f_{\{1, \dots, k\}, S_2}(\ell) \\ \times \left(\prod_{i=1}^k |a_i|^{-\frac{1}{k}} \right) \prod_{\substack{i=1, \\ i \notin S_1 \cup S_2}}^k \left(2\gamma - \log(|a_i| \ell^2 / (\ell, a_i)^2) + \frac{1}{k} \sum_{r=1}^k \log |a_r| \right) \end{aligned} \quad (3.3)$$

(we remark that $f_{\{1, \dots, k\}, S_2}(\ell) / \varphi(\ell)$ does not depend on ℓ). Among these, the term with $m = k$ is

$$\frac{k!}{(k(s-1) + 1)^{k+1}} \frac{\Delta_{\mathbf{0}; \{1, \dots, k\}} \Gamma(\frac{1}{k})^k}{|a_1 \cdots a_k|^{\frac{1}{k}}} \sum_{\ell \geq 1} \frac{\prod_{i=1}^k (a_i, \ell)}{\ell^k} \varphi(\ell) \quad (3.4)$$

since $f_{\{1, \dots, k\}, \emptyset} = \varphi(\ell) \Gamma(\frac{1}{k})^k \Delta_{\mathbf{0}; \{1, \dots, k\}}$. Finally,

$$\begin{aligned} \sum_{\ell \geq 1} \frac{\prod_{i=1}^k (a_i, \ell)}{\ell^k} \varphi(\ell) &= \rho(\mathbf{a}) \prod_p \left(1 + \sum_{m=1}^{\infty} \left(1 - \frac{1}{p}\right) p^{-m(k-1)} \right) = \rho(\mathbf{a}) \prod_p \left(1 + \frac{p-1}{p^k - p} \right) \\ &= \rho(\mathbf{a}) \frac{\zeta(k-1)}{\zeta(k)}, \end{aligned}$$

where

$$\rho(\mathbf{a}) := \prod_{p|a_1 \cdots a_k} \frac{1 + \sum_{m=1}^{\infty} \left(1 - \frac{1}{p}\right) p^m \prod_{i=1}^k p^{-\max(0, m - \nu_p(a_i))}}{1 + (p-1)/(p^k - p)} \quad (3.5)$$

with $\nu_p(a) = r$ if $p^r \parallel a$. Thus, the expression in (3.4) can be rewritten as

$$\frac{\rho(\mathbf{a})}{|a_1 \cdots a_k|^{\frac{1}{k}}} \frac{k!}{(k(s-1)+1)^{k+1}} \frac{\zeta(k-1)\Gamma(\frac{1}{k})^k}{\zeta(k)\Gamma(\frac{r}{k})\Gamma(1-\frac{r}{k})},$$

where $r = \{1 \leq i \leq k \mid \text{sign}(a_i) = 1\}$ (and the above expression has to be interpreted as 0 if $r \in \{0, k\}$).

We remark that for all $q \geq 1$ one has, as expected, $\rho(qa_1, \dots, qa_k) = q\rho(a_1, \dots, a_k)$. Finally, we observe that if we assume $\text{GCD}(a_1, \dots, a_k) = 1$ and let $\kappa(\mathbf{a}) := \prod_{p \mid a_1 \cdots a_k} p^{m_p}$, where m_p is the second smallest among $\nu_p(a_1), \dots, \nu_p(a_k)$ (the smallest being 1 by hypothesis), then we have

$$\frac{\zeta(k)}{\zeta(k-1)} \frac{\varphi(\kappa(\mathbf{a}))}{\kappa(\mathbf{a})} d(\kappa(\mathbf{a})) \leq \rho(\mathbf{a}) \leq d(\kappa(\mathbf{a})) \quad (3.6)$$

and so, in particular $1 \ll \rho(\mathbf{a}) \ll_\varepsilon |a_1 \cdots a_k|^\varepsilon$. Indeed, we have

$$1 + \sum_{m=1}^{n_p} \left(1 - \frac{1}{p}\right)^m \leq 1 + \sum_{m=1}^{\infty} \left(1 - \frac{1}{p}\right)^m p^m \prod_{i=1}^k p^{-\max(0, m - \nu_p(a_i))} \leq 1 + \sum_{m=1}^{n_p} \left(1 - \frac{1}{p}\right)^m + \sum_{m > n_p} \left(1 - \frac{1}{p}\right)^m p^{n_p - m}$$

and so

$$(n_p + 1) \left(1 - \frac{1}{p}\right) \leq 1 + \sum_{m=1}^{\infty} \left(1 - \frac{1}{p}\right)^m p^m \prod_{i=1}^k p^{-\max(0, m - \nu_p(a_i))} \leq 1 + n_p \left(1 - \frac{1}{p}\right) + \frac{1}{p} \leq n_p + 1$$

which gives

$$\frac{\prod_{p \mid \kappa(\mathbf{a})} (v_p(\kappa(\mathbf{a})) + 1) \left(1 - \frac{1}{p}\right)}{1 + (p-1)/(p^k - p)} \leq \rho(\mathbf{a}) \leq \prod_{p \mid \kappa(\mathbf{a})} (v_p(\kappa(\mathbf{a})) + 1)$$

and so (3.6) follows.

Remark 4. Also the coefficients $c_{k-1, r}$ with $1 \leq r \leq k$ can be expressed in terms of the Gamma and zeta functions. Indeed, by Ramanujan's formula (1.10) if $\Re(s) > 1$, $\Re(s+w) > 1$, then

$$\begin{aligned} \sum_{\ell \geq 1} \frac{1}{\ell^w} \sum_{h \pmod{\ell}}^* D_{0,0}(s, \pm h/k) &= \sum_{\ell \geq 1} \frac{1}{\ell^w} \sum_{n \geq 1} \frac{d(n)}{n^s} c_\ell(n) = \frac{1}{\zeta(w)} \sum_{n \geq 1} \frac{d(n) \sigma_{1-w}(n)}{n^s} \\ &= \frac{\zeta(s)^2 \zeta(s+w-1)^2}{\zeta(w) \zeta(2s+w-1)}, \end{aligned}$$

since $\sum_{n \geq 1} \frac{\sigma_a(n) \sigma_b(n)}{n^s} = \frac{\zeta(s) \zeta(s-a) \zeta(s-b) \zeta(s-a-b)}{\zeta(2s-a-b)}$ (cf. (1.3.3) in [Tit]) and the same formula holds for $\Re(s+w) > 2$, $\Re(2s+w) > 2$, $\Re(w) > 1$ by analytic continuation. In particular, assuming

for simplicity $\mathbf{a} = (-1, 1, \dots, 1)$ one computes that the contribution of the terms with $|\mathcal{I}| = k - 1$ to (3.1) is

$$(k-1) \left(\prod_{i=1}^{k-1} \frac{1}{2\pi i} \int_{|w_i|=\frac{\varepsilon}{k}} \zeta(1+w_i)^2 \Gamma(w_i + \frac{1-s_*}{k-1}) \right) \frac{\Gamma(1-w_1 - \frac{1-s_*}{k-1})^{-1} \Gamma(w_1 + \frac{1-s_*}{k-1})^{-1}}{(k-1)(s-1) - s_* + 1} \\ \times \frac{\zeta(1 - \frac{1}{k-1} + \frac{s_*}{k-1})^2 \zeta(k - \frac{k}{k-1} + \frac{(2k-1)s_*}{k-1})^2}{\zeta(k-1 + 2s_*) \zeta(k - \frac{2}{k-1} + \frac{2ks_*}{k-1})} \prod_{i=1}^{k-1} dw_i,$$

where $s_* = s_*(w_1, \dots, w_{k-1}) := \sum_{r=1}^{k-1} w_r$. Proceeding as above one can then compute the coefficients $c_{k-1,r}$ for $1 \leq r \leq k$.

4 The proof of Theorem 2

First, we observe that the contribution to $N(B^k)$ coming from the terms where the maximum $\max\{|x_1|, \dots, |x_k|, |y_1|, \dots, |y_k|\}$ is attained at more than one of the $|x_i|, |y_j|$ is $O_{\mathbf{a}, \varepsilon}(B^{k-\frac{3}{2}+\varepsilon})$. Indeed, the contribution of the terms with $|x_i|, |y_i| \leq |x_1| = |y_2|$ for all $i = 1, \dots, k$ is

$$\sum_{\substack{x_i, y_i \leq x_1 = y_2 < \sqrt{B}, \forall i=1, \dots, k \\ a_1 x_1 y_1 + \dots + a_k x_k y_k = 0}} 1 = \sum_{\substack{x_i, y_i \leq x_1 = y_2 < \sqrt{B}, \forall i=1, \dots, k \\ x_1(a_1 y_1 + a_2 x_2) = a_3 x_3 y_3 + \dots + a_k x_k y_k}} 1 \ll \sum_{\substack{r < (|a_1| + |a_2|)B, \\ x_i, y_i < \sqrt{B}, \forall i=3, \dots, k \\ r = a_3 x_3 y_3 + \dots + a_k x_k y_k}} d(r) B^{\frac{1}{2}} \ll_{\mathbf{a}, \varepsilon} B^{k-\frac{3}{2}+\varepsilon}.$$

and one can bound similarly all the other cases. Thus,

$$N(B^k) = \sum_{i=1}^k N_i(B) + O_{\mathbf{a}, \varepsilon}(B^{k-\frac{3}{2}+\varepsilon}),$$

where

$$N_i(B) = \frac{1}{2} \# \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{\neq 0}^{2k} \left| \begin{array}{l} a_1 x_1 y_1 + \dots + a_k x_k y_k = 0, \quad (x_1, \dots, x_k) = (y_1, \dots, y_k) = 1 \\ |x_i y_j| < B \quad \forall j = 1, \dots, k, |x_{j_1}|, |y_{j_2}| < |x_i| \quad \forall j_1, j_2 = 1, \dots, k, j_1 \neq i \end{array} \right. \right\} \\ = 2^{k-1} \# \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{N}^k \times \mathbb{Z}_{\neq 0}^k \left| \begin{array}{l} a_1 x_1 y_1 + \dots + a_k x_k y_k = 0, \quad (x_1, \dots, x_k) = (y_1, \dots, y_k) = 1 \\ |x_i y_j| < B \quad \forall j = 1, \dots, k, |y_{j_2}|, x_{j_1} < x_i \quad \forall j_1, j_2 = 1, \dots, k, j_1 \neq i \end{array} \right. \right\}.$$

Notice that in the first line we divided by 4 since $-\mathbf{x} = \mathbf{x}$ in $\mathbb{P}^1(\mathbb{Q})$ and we multiplied by 2 since we assumed the maximum among the x_i, y_j is attained at one of the x_i .

By symmetry it is sufficient to consider the case $i = 1$. Also, we can assume B is a half-integer. Using Möbius inversion formula we find

$$N_1(B) = 2^{k-1} \sum_{d_1, d_2 \geq 1} \frac{\mu(d_1) \mu(d_2)}{d_1 d_2} \sum_{\substack{x_2, \dots, x_k, |y_1|, \dots, |y_k| < x_1, \quad x_i \in \mathbb{N}, y_i \in \mathbb{Z}_{\neq 0} \\ x_1 |y_j| < B/d_1 d_2, \quad \forall j=1, \dots, k, \\ a_1 x_1 y_1 + \dots + a_k x_k y_k = 0}} 1.$$

Then, we express the inequalities $x_1|y_j| < B/d_1d_2$ and $x_2 \dots, x_k, |y_1|, \dots, |y_k| < x_1$ analytically via the following formula (see Theorem G in [Ing])

$$\frac{1}{2\pi i} \int'_{(\delta)} x^{-z} \frac{dz}{z} = \begin{cases} \chi_{(0,1)}(x) + O_\delta\left(\frac{x^\delta}{T|\log x|}\right) & \text{if } x \in \mathbb{R}_{>0} \setminus \{1\} \\ 1/2 + O_\delta(T^{-1}) & \text{if } x = 1 \end{cases}$$

where $\chi_{[0,1]}(x)$ is the indicator function of the set $[0, 1)$ and \int' indicates that the integral is truncated at $|\Im(z)| \leq T$. We shall choose the parameter $1 \leq T \leq B$ at the end of the argument. Bounding as above the error coming from the cases where $x_1/x_i = 1$, $x_1/y_j = 1$ for some $i = 2, \dots, k$ or $j = 1, \dots, k$, we obtain

$$\begin{aligned} N_1(B) &= \sum_{d_1, d_2 \geq 1} \frac{\mu(d_1)\mu(d_2)}{d_1d_2} \sum_{\substack{x_1, \dots, x_k \in \mathbb{N}, y_1, \dots, y_k \in \mathbb{Z}_{\neq 0}, \\ a_1x_1y_1 + \dots + a_kx_ky_k = 0}} \left(\prod_{j=1}^k \frac{1}{(2\pi i)^2} \int'_{(c_{z_j}, c_{w_j})} \frac{1}{y_j^{z_j+w_j}} \right) \\ &\times \left(\prod_{f=2}^k \frac{1}{2\pi i} \int''_{(c_{u_f})} \frac{1}{x_f^{u_f}} \right) \frac{2^{k-1}(B/d_1d_2)^{\sum_{j=1}^k z_j}}{x_1^{\sum_{j=1}^k (z_j-w_j) - \sum_{f=2}^k u_f}} \left(\prod_{j=1}^k \frac{dz_j dw_j}{z_j w_j} \right) \left(\prod_{f=2}^k \frac{du_f}{u_f} \right) + E \end{aligned} \quad (4.1)$$

where \int'' indicates the integral is truncated at $|\Im(u_f)| \leq 2T$ and the lines of integration are $c_{z_j} = 1 - \frac{1}{k} + 3\varepsilon$, $c_{w_j} = \varepsilon$ and $c_{u_f} = 1 - \frac{1}{k} + \varepsilon$, for some small $\varepsilon > 0$. The error term E is $O_{\varepsilon, \mathbf{a}, k}(B^{k-1+3k\varepsilon}/T)$. Indeed, for example, in the most delicate case one needs to bound sums of the following form (we take $a_1 = \dots = a_k = 1$ for simplicity, but the same proof extends to the general case):

$$\sum_{\substack{x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{Z}_{\neq 0}, x_1 \neq y_2 \\ x_1y_1 + \dots + x_ky_k = 0}} \frac{B^{k-1+3k\varepsilon}}{|x_1 \cdots x_k y_1 \cdots y_k|^{1-\frac{1}{k}+\varepsilon}} \frac{1}{T \log |x_1/y_2|}. \quad (4.2)$$

Now, $(\log |x/y|)^{-1} \ll \frac{|x|}{|x-y|}$ if $|x-y| < |x|/2$ (and thus $|y| \geq |x|/2$) and $(\log |x/y|)^{-1} \ll 1$ otherwise. Thus, this sum is

$$\ll_{k, \varepsilon} \frac{B^{k-1+3k\varepsilon}}{T} + \sum_{\substack{x_1, \dots, x_k, y_1, y_2, \dots, y_k \in \mathbb{Z}_{\neq 0}, 0 \neq |x_1-y_2| < |x_1|/2 \\ x_1y_1 + x_2y_2 + \dots + x_ky_k = 0}} \frac{B^{k-1+3k\varepsilon}/T}{|x_2 \cdots x_k y_1 y_3 \cdots y_k|^{1-\frac{1}{k}+\varepsilon}} \frac{1}{|x_1 - y_2| |x_1|^{1-\frac{2}{k}+\varepsilon}}.$$

Now, a simple computation shows that for $m \neq 0$

$$\sum_{m=x_3y_3 + \dots + x_ky_k} \frac{1}{|x_3 \cdots x_k y_3 \cdots y_k|^{1-\frac{1}{k}+\varepsilon}} \ll_\varepsilon m^{-\frac{2}{k}}$$

and so, writing $r = x_1 - y_2$, $n = y_1 + x_2$ and $m = x_3y_3 + \cdots + x_ky_k$ and bounding easily the case $m = 0$, we see that the sum in (4.2) is bounded by

$$\ll_{k,\varepsilon} \frac{B^{k-1+3k\varepsilon}}{T} + \frac{B^{k-1+3k\varepsilon}}{T} \sum_{\substack{x_1, m, r, x_2 \in \mathbb{Z} \neq 0, n \in \mathbb{Z} \\ x_1 n - x_2 r = m, \\ |r| < |x_1|/2, n \neq x_2}} \frac{1}{|x_1|^{1-\frac{2}{k}+\varepsilon} |n-x_2|^{1-\frac{1}{k}+\varepsilon} |x_2|^{1-\frac{1}{k}+\varepsilon} |r| |m|^{\frac{2}{k}}}.$$

The terms with $|m| > (|x_1 n| + |x_2 r|)/4$ can be bounded easily by using $|m|^{-\frac{2}{k}} \ll |x_1 n|^{-\frac{2}{k}}$ and disregarding the linear equation. For the terms with $|m| \leq (|x_1 n| + |x_2 r|)/4$, we have also $|x_1 n| \leq \frac{5}{3}|x_2 r|$ and so, since $r < |x_1|/2$, then $|n| < \frac{5}{6}|x_2|$ and so $|n-x_2|^{1-\frac{1}{k}+\varepsilon} \gg |x_2|^{1-\frac{1}{k}+\varepsilon}$. Thus, we obtain that the sum in (4.2) is

$$\ll_{k,\varepsilon} \frac{B^{k-1+3k\varepsilon}}{T} + \frac{B^{k-1+3k\varepsilon}}{T} \sum_{\substack{x_1, m, r, x_2, n \in \mathbb{Z} \neq 0 \\ x_1 n - x_2 r = m, \\ m \ll |x_2 r|, r \ll |x_1|}} \frac{1}{|x_1|^{1-\frac{2}{k}+\varepsilon} |x_2|^{2-\frac{2}{k}+\varepsilon} |r| |m|^{\frac{2}{k}}}.$$

Then, we write $m = \ell + x_1 g$ with $\ell \equiv -x_2 r \pmod{|x_1|}$, $|g| \ll \frac{|x_2 r|}{|x_1|}$, $-\frac{|x_1|}{2} < \ell \leq \frac{|x_1|}{2}$ and $(\ell, g) \neq (0, 0)$, $\ell + x_1 g \neq x_2 r$. Dividing according to whether $g \neq 0$ and $g = 0$ we obtain that the above sum is

$$\begin{aligned} &\ll \sum_{\substack{x_1, r, x_2, g \in \mathbb{Z} \neq 0 \\ |g| \ll |x_2 r|/|x_1|, \\ r \ll |x_1|}} \frac{B^{k-1+3k\varepsilon}/T}{|x_1|^{1-\frac{2}{k}+\varepsilon} |x_2|^{2-\frac{2}{k}+\varepsilon} |r| |g x_1|^{\frac{2}{k}}} + \sum_{\substack{x_1, r, x_2 \in \mathbb{Z} \neq 0 \\ 1 \leq |\ell| \leq \frac{|x_1|}{2}, r \ll |x_1|, \\ x_1 |(\ell - x_2 r)| \neq 0}} \frac{B^{k-1+3k\varepsilon}/T}{|x_1|^{1-\frac{2}{k}+\varepsilon} |x_2|^{2-\frac{2}{k}+\varepsilon} |r| |\ell|^{\frac{2}{k}}} \\ &\ll \sum_{\substack{x_1, r, x_2 \in \mathbb{Z} \neq 0 \\ r \ll |x_1|}} \frac{B^{k-1+3k\varepsilon}/T}{|x_1|^{2-\frac{2}{k}+\varepsilon} |x_2|^{1+\varepsilon} |r|^{\frac{2}{k}}} + \sum_{\substack{\ell, r, x_2 \in \mathbb{Z} \neq 0 \\ \ell - x_2 r \neq 0}} \frac{d(\ell - x_2 r) B^{k-1+3k\varepsilon}/T}{|x_2|^{2-\frac{2}{k}+\varepsilon} |r|^{1+\frac{\varepsilon}{2}} |\ell|^{1+\frac{\varepsilon}{2}}} \ll \frac{B^{k-1+3k\varepsilon}}{T}, \end{aligned}$$

as claimed.

Now, we go back to (4.1) and make the change of variables $u_f \rightarrow u_f + z_f$ for all $f = 2, \dots, k$. Summing the Dirichlet series, we obtain

$$\begin{aligned} N_1(B) &= 2^{k-1} \left(\prod_{j=1}^k \frac{1}{(2\pi i)^2} \int'_{(c_{w_j}, c_{z_j})} \right) \left(\prod_{f=2}^k \frac{1}{2\pi i} \int'''_{(c_{u_f})} \right) \frac{B^{\sum_{j=1}^k z_j} \mathcal{A}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \mathbf{w}}^*(0)}{\zeta(1 + \sum_{j=1}^k z_j)^2} \\ &\quad \times \left(\prod_f \frac{du_f}{u_f + z_f} \right) \left(\prod_j \frac{dz_j dw_j}{z_j w_j} \right) + O_{k,\varepsilon, \mathbf{a}}(B^{k-1+\varepsilon}/T), \end{aligned}$$

where $\mathcal{A}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \boldsymbol{\beta}}^*(0)$ is as defined in (2.7), $c_{u_f} = -2\varepsilon$,

$$\boldsymbol{\alpha} := (-w_1 - \sum_{j=2}^k (w_j + u_j), u_2, \dots, u_k)$$

and \int''' indicates the integral is truncated at $|\Im(z_f + u_f)| \leq 2T$.

Now, we apply Theorem 3 to $\mathcal{A}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \boldsymbol{\beta}}^*(0) = \mathcal{A}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha} - \boldsymbol{\xi}, \mathbf{z} + \boldsymbol{\beta} - \boldsymbol{\xi}}^*(1 - \frac{1}{k})$, where $\boldsymbol{\xi} := (1 - \frac{1}{k}, \dots, 1 - \frac{1}{k})$. We keep as main term only the summand in (2.9) with $\mathcal{I} = \{1, \dots, k\}$, treating the other summands as error terms. Thus, we write

$$\mathcal{A}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \mathbf{w}}^*(0) = \mathcal{M}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \mathbf{w}}^{**} + \mathcal{E}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \mathbf{w}}^{**},$$

where

$$\begin{aligned} \mathcal{M}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \mathbf{w}}^{**} &= \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i, w_i\} \\ \forall i=1, \dots, k}} \left(\prod_{i=1}^k \frac{\zeta(1 - \alpha_i^* + \beta_i^*)}{|a_i|^{-z_i - \alpha_i^* + \frac{1 + \sum_{r=1}^k (z_r + \alpha_r^*)}{k}}} \frac{\Gamma(-\frac{z_i + \alpha_i^*}{2} + \frac{1 + \sum_{r=1}^k (z_r + \alpha_r^*)}{2k})}{\Gamma(\frac{1 + z_i + \alpha_i^*}{2} - \frac{1 + \sum_{r=1}^k (z_r + \alpha_r^*)}{2k})} \right) \\ &\times \frac{2\pi^{\frac{k}{2} - 1}}{1 + \sum_{i=1}^k (z_i + \alpha_i^* - 1)} \sum_{\ell \geq 1} \frac{\prod_{i=1}^k (a_i, \ell)^{1 - \alpha_i^* + \beta_i^*}}{\ell^{\sum_{i=1}^k (1 - \alpha_i^* + \beta_i^*)}} \varphi(\ell), \end{aligned}$$

and $\mathcal{E}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \boldsymbol{\beta}}^{**}$ is holomorphic on a region containing

$$\eta := \sum_{i=1}^k (|\Re(z_i + \alpha_i - 1 + \frac{1}{k})| + |\Re(z_i + w_i - 1 + \frac{1}{k})|) < \frac{1 - \varepsilon}{2k},$$

where it satisfies

$$\mathcal{E}_{\mathbf{a}; \mathbf{z} + \boldsymbol{\alpha}, \mathbf{z} + \boldsymbol{\beta}}^{**} \ll_{k, \varepsilon, \mathbf{a}} \left((1 + \max_{i=1}^k (|\Im(\alpha_i + z_i)| + |\Im(z_i + w_i)|)) \right)^{14k(\eta + \varepsilon)}.$$

Assuming ε is small enough with respect to k , we can bound the contribution coming from \mathcal{E}^{**} by moving the line of integration c_{z_k} to $c_{z_k} = \frac{1}{2k} - 6k\varepsilon$, obtaining a contribution of $O(B^{k-1+26k\varepsilon}(T^7 B^{-\frac{1}{2k}} + T^{-1}))$ from the integrals over the new line of integration and on the horizontal segments. Thus, we obtain

$$N_1(B) = N_1'(B) + N_1''(B) + O_{k, \varepsilon, \mathbf{a}}(B^{k-1+26k\varepsilon}(T^7 B^{-\frac{1}{2k}} + T^{-1})),$$

where

$$\begin{aligned} N_1'(B) &:= \left(\prod_{j=1}^k \frac{1}{(2\pi i)^2} \int'_{(c_{w_j}, c_{z_j})} \right) \left(\prod_{f=2}^k \frac{1}{2\pi i} \int'''_{(c_{u_f})} \right) \sum_{\substack{(\alpha_1^*, \beta_1^*) = (\alpha_1, w_1) \\ \{\alpha_i^*, \beta_i^*\} = \{u_i, w_i\} \\ \forall i=2, \dots, k}} Q_{\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*}(\mathbf{z}, \mathbf{w}) \\ &= \frac{\prod_{i=1}^k \zeta(1 - \alpha_i^* + \beta_i^*)}{1 + \sum_{i=1}^k (z_i + \alpha_i^* - 1)} \frac{B^{\sum_{j=1}^k z_j}}{\zeta(1 + \sum_{j=1}^k z_j)^2} \left(\prod_{f=2}^k du_f \right) \left(\prod_{j=1}^k \frac{dz_j dw_j}{w_j} \right) \end{aligned}$$

with

$$\begin{aligned} \mathcal{Q}_{\alpha^*, \beta^*}(\mathbf{z}, \mathbf{w}) &:= \left(\prod_{i=1}^k \frac{1}{|a_i|^{-z_i - \alpha_i^* + \frac{1 + \sum_{r=1}^k (z_r + \alpha_r^*)}{k}}} \frac{\Gamma(-\frac{z_i + \alpha_i^*}{2} + \frac{1 + \sum_{r=1}^k (z_r + \alpha_r^*)}{2k})}{\Gamma(\frac{1 - z_i - \alpha_i^*}{2} + \frac{1 + \sum_{r=1}^k (z_r + \alpha_r^*)}{2k})} \right) \\ &\times \frac{2^k \pi^{\frac{k}{2} - 1}}{z_1 \prod_{f=2}^k z_f (z_f + u_f)} \sum_{\ell \geq 1} \frac{\prod_{i=1}^k (a_i, \ell)^{1 - \alpha_i^* + \beta_i^*}}{\ell^{\sum_{i=1}^k (1 - \alpha_i^* + \beta_i^*)}} \varphi(\ell), \end{aligned}$$

and where $N''_{1, \varepsilon}(B)$ is defined in the same way, but with the condition $(\alpha_1^*, \beta_1^*) = (\alpha_1, w_1)$ in the sum replaced by $(\alpha_1^*, \beta_1^*) = (w_1, \alpha_1)$. We remark that if ε is small enough, $\mathcal{Q}_{\alpha^*, \beta^*}(\mathbf{z}, \mathbf{w})$ is holomorphic on a region containing

$$\begin{aligned} \sum_{i=1}^k \Re(\beta_i^* - \alpha_i^*) &> 2 - k + \varepsilon, \quad \Re(z_f + u_f) > 0 \quad \forall f = 2, \dots, k, \\ 1 - \frac{3}{2k} &\leq \Re(z_i + \alpha_i^*) < 1 - \frac{1}{k} + 2k\varepsilon, \quad \Re(z_i) > 0 \quad \forall i = 1, \dots, k, \end{aligned}$$

where by Stirling's formula it satisfies

$$\begin{aligned} \mathcal{Q}_{\alpha^*, \beta^*}(\mathbf{z}, \mathbf{w}) &\ll_{\mathbf{a}, k, \varepsilon} \frac{\prod_{i=1}^k |\Im(z_i + \alpha_i^* - \frac{1}{k} \sum_{r=1}^k (z_r + \alpha_r^*))|^{-\frac{1}{2} - \Re(z_i + \alpha_i^* - \frac{1 + \sum_{r=1}^k (z_r + \alpha_r^*)}{k})}}{|z_1| \prod_{i=2}^k |z_i(z_i + u_i)|} \\ &\ll_{\mathbf{a}, k, \varepsilon} \frac{\prod_{i=1}^k |\Im(z_i + \alpha_i^* - \frac{1}{k} \sum_{r=1}^k (z_r + \alpha_r^*))|^{\frac{1}{2} - \Re(z_i + \alpha_i^*) + \varepsilon}}{|z_1| \prod_{i=2}^k |z_i(z_i + u_i)|}. \end{aligned} \quad (4.3)$$

Now we move c_{z_1} to $c_{z_1} = 1 - \frac{3}{2k} + 6k\varepsilon$ passing through the pole at $k - 1 - \sum_{i=1}^k (z_i + \alpha_i^*) = 0$. Notice that doing so $\alpha_1^* - \beta_1^*$ stays constant so we don't cross the pole of $\zeta(1 - \alpha_1^* + \beta_1^*)$. The contribution of the integrals on the new line of integration and on the horizontal segments is trivially $O_{k, \varepsilon, \mathbf{a}}(B^{k-1+12k\varepsilon}(B^{-\frac{1}{2k}} + T^{-1}))$, whereas the contribution of the residue is

$$\begin{aligned} &\frac{1}{2\pi i} \int'_{(c_{w_1})} \left(\prod_{j=2}^k \frac{1}{(2\pi i)^2} \int'_{(c_{w_j})} \int'_{(c_{z_j})} \right) \left(\prod_f \frac{1}{2\pi i} \int'_{(\delta_{c_{u_f}})} \right) \sum_{\substack{(\alpha_1^*, \beta_1^*) = (\alpha_1, \beta_1) \\ \{\alpha_i^*, \beta_i^*\} = \{u_i, w_i\} \\ \forall i=2, \dots, k}} \mathcal{Q}_{\alpha^*, \beta^*}(\mathbf{z}', \mathbf{w}) \\ &\left(\prod_{i=1}^k \zeta(1 - \alpha_i^* + \beta_i^*) \right) \frac{B^{k-1+w_1+\sum_{j=2}^k \beta_j^*}}{\zeta(k+w_1+\sum_{j=2}^k \beta_j^*)^2} \left(\prod_f du_f \right) \left(\prod_{j=2}^k \frac{dz_j dw_j}{w_j} \right) \frac{dw_1}{w_1} \end{aligned}$$

where $\mathbf{z}' = (k - 1 - \alpha_1 - \sum_{i=2}^k (z_i + \alpha_i^*), z_2, \dots, z_k)$ and where we used that for $z_1 = k - 1 - \alpha_1^* - \sum_{i=2}^k (z_i + \alpha_i^*)$ one has

$$\sum_{j=1}^k z_j = k - 1 - \sum_{j=1}^k \alpha_j^* = k - 1 + w_1 - \sum_{j=2}^k \alpha_j^* + \sum_{j=2}^k (w_j + u_j) = k - 1 + w_1 + \sum_{j=2}^k \beta_j^*.$$

Next, we observe that the terms in the sum over α^*, β^* for which $(\alpha_h^*, \beta_h^*) = (w_h, u_h)$ for some $h \in \{2, \dots, k\}$ are $O_{k,\varepsilon,\alpha}(B^{k-1+12k\varepsilon}(TB^{-\frac{1}{2k}} + T^{-1}))$; indeed one can move c_{w_h} and c_{u_h} to $c_{u_h} = -\frac{1}{2k} + (6k-2)\varepsilon$ and $c_{w_h} = \frac{1}{2k} - (6k-1)\varepsilon$ and then bound trivially obtaining the claimed bound. Notice that doing so we don't pass through any poles, since $\Re(1 - \alpha_1^* + \beta_1^*) = 1 + \Re(2w_1 + \sum_{j=2}^k u_j + w_j)$ stays constant, whereas $\Re(1 - \alpha_h^* + \beta_h^*) = \Re(1 - w_h^* + u_h^*)$ stays less than one.

Thus, we only have to consider the term with $(\alpha_j^*, \beta_j^*) = (u_j, w_j)$ for all $j = 2, \dots, k$ and moving the lines of integration as above for all $i = 2, \dots, k$ one obtains that it's enough to consider the contribution from the residue at $w_j = 0$ for all $j = 2, \dots, k$. To summarize, we arrive to

$$N'_1(B) = \frac{1}{2\pi i} \int'_{(c_{w_1})} \left(\prod_{j=2}^k \frac{1}{(2\pi i)^2} \int'_{(c_{z_j})} \int'''_{(c_{u_j})} \right) \sum_{\substack{(\alpha_1^*, \beta_1^*) = (-w_1 - \sum_{j=2}^k u_j, w_1) \\ (\alpha_i^*, \beta_i^*) = (u_i, 0) \\ \forall i=2, \dots, k}} Q_{\alpha^*, \beta^*}(\mathbf{z}', \mathbf{w}') B^{k-1+w_1} \\ \times \zeta(1 + 2w_1 + \sum_{j=2}^k u_j) \frac{\prod_{i=2}^k \zeta(1 - u_i)}{\zeta(k + w_1)^2} \frac{dw_1}{w_1} \prod_{j=2}^k du_j dz_j + O_{k,\varepsilon,\alpha} \left(B^{k-1+12k\varepsilon} \left(\frac{T^{Ak}}{B^{-\frac{1}{2k}}} + \frac{1}{T} \right) \right)$$

with $\mathbf{w}' = (w_1, 0, \dots, 0)$. Next we move the line of integration c_{w_1} to $c_{w_1} = -\frac{1}{4k} + 6k\varepsilon$ passing through the pole at $w_1 = 0$ only, so that bounding trivially the contribution of the new line of integration we obtain $N'_1(B) = C_{1,1}(T)B^{k-1} + O_{k,\varepsilon,\alpha}(B^{k-1+12k\varepsilon}(TB^{-\frac{1}{4k}} + T^{-1}))$, where

$$C_{1,1}(T) := \left(\prod_{j=2}^k \frac{1}{(2\pi i)^2} \int'_{(c_{z_j})} \int'''_{(c_{u_j})} \right) \frac{Q_{\mathbf{w}', \mathbf{0}}(\mathbf{z}', \mathbf{0})}{\zeta(k)^2} \zeta(1 + \sum_{j=2}^k u_j) \prod_{j=2}^k \zeta(1 - u_j) du_j dz_j$$

with $\mathbf{u}' = (-\sum_{j=2}^k u_j, u_2, \dots, u_k)$ and $\mathbf{z}' = (k-1 - \alpha_1^* - \sum_{i=2}^k (z_i + u_i), z_2, \dots, z_k) = (k-1 - \sum_{i=2}^k z_i, z_2, \dots, z_k)$ and lines of integration $c_{u_j} = -2\varepsilon$, $c_{z_i} = 1 - \frac{1}{k} + 3\varepsilon$. Notice that (4.3) in this case gives

$$Q_{\alpha^*, \beta^*}(\mathbf{z}', \mathbf{0}) \ll_{\mathbf{a}, k, \varepsilon} \frac{\prod_{i=2}^k |\Im(z_i + u_i)|^{-\frac{1}{2} + \frac{1}{k}}}{|k-1 - z_2 + \dots + z_k| \prod_{i=2}^k |z_i(z_i + u_i)|}$$

and thus, using the convexity bound $\zeta(1 + \sum_{j=2}^k u_j) \ll_\varepsilon (1 + |u_2 + \dots + u_k|)^{(k+1)\varepsilon}$, we obtain

$$C_{1,1}(T) := C'_{1,1} + O(T^{-\frac{1}{2} + \frac{1}{k} + 2k\varepsilon}),$$

where $C'_{1,1}$ is defined as $C_{1,1}(T)$ but where we removed the truncations at $|\Im(z_i)| \leq T$ and $|\Im(z_i + u_i)| \leq 2T$ from the integrals. Thus,

$$N'_1(B) = C'_{1,1} B^{k-1} + O_{k,\varepsilon,\alpha}(B^{k-1+12k\varepsilon}(TB^{-\frac{1}{4k}} + T^{-\frac{1}{2} + \frac{1}{k}})).$$

We can treat $N_1''(B)$ in the same way, the only difference being that in this case $\sum_j z_j = k - 1 - w_1 - \sum_{j=2}^k \alpha_j^*$ so that we still obtain a non-negligible contribution only from the summand with $(\alpha_i^*, \beta_i^*) = (u_i, w_i)$ for all $i = 2, \dots, k$. We arrive to

$$N_1''(B) = \frac{1}{2\pi i} \int'_{(c_{w_1})} \left(\prod_{j=2}^k \frac{1}{(2\pi i)^2} \int'_{(c_{z_j})} \int''''_{(c_{u_j})} \right) \sum_{\substack{(\alpha_1^*, \beta_1^*) = (w_1, -w_1 - \sum_{j=2}^k u_j) \\ (\alpha_i^*, \beta_i^*) = (u_i, 0) \quad \forall i=2, \dots, k}} Q_{\alpha^*, \beta^*}(\mathbf{z}', \mathbf{w}') B^{k-1-w_1-\sum_{j=2}^k u_j} \\ \frac{\zeta(1-2w_1-\sum_{j=2}^k u_j)}{\zeta(k-w_1-\sum_{j=2}^k u_j)^2} \left(\prod_{j=2}^k \zeta(1-u_j) du_j dz_j \right) \frac{dw_1}{w_1} + O_{k,\varepsilon,\mathbf{a}} \left(B^{k-1+12k\varepsilon} \left(TB^{-\frac{1}{2k}} + T^{-1} \right) \right),$$

with $\mathbf{z}' = (k-1+w_1-\sum_{j=2}^k u_j - \sum_{j=2}^k z_j, z_2, \dots, z_k)$ and $\mathbf{w}' = (w_1, 0, \dots, 0)$. We move the line of integration c_{w_1} to $c_{w_1} = \frac{1}{4k} - 6k\varepsilon$, passing through a pole at $w_1 = -\frac{1}{2} \sum_{j=2}^k u_j$. The integral on the new line of integration can be bounded trivially, whereas the contribution of the residue is

$$\left(\prod_{j=2}^k \frac{1}{(2\pi i)^2} \int'_{(c_{z_j})} \int''''_{(c_{u_j})} \right) \sum_{\substack{(\alpha_1^*, \beta_1^*) = (-\frac{1}{2} \sum_{j=2}^k u_j, -\frac{1}{2} \sum_{j=2}^k u_j) \\ (\alpha_i^*, \beta_i^*) = (u_i, 0) \quad \forall i=2, \dots, k}} Q_{\alpha^*, \beta^*}(\mathbf{z}, \mathbf{w}') \\ \times \frac{-1}{\sum_{j=2}^k u_j \zeta(k - \frac{1}{2} \sum_{j=2}^k u_j)^2} \left(\prod_{j=2}^k \zeta(1+u_j) du_j dz_j \right)$$

where $\mathbf{w}' = (-\frac{1}{2} \sum_{j=2}^k u_j, 0, \dots, 0)$. Next, for each $j = 3, \dots, k$ we move c_{u_j} to $c_{u_j} = \frac{1}{2k} - 6k\varepsilon$, passing through the pole of $\zeta(1-u_j)$. The contribution on the new line of integration can be bounded trivially, and we obtain that the above is

$$\frac{-1}{2\pi i} \int'_{(c_{u_2})} \left(\prod_{j=2}^k \frac{1}{2\pi i} \int'_{(c_{z_j})} \right) Q_{\alpha', \beta'}(\mathbf{z}, \mathbf{w}') \frac{\zeta(1-u_2)}{u_2} \frac{B^{k-\frac{1}{2}u_2}}{\zeta(k-\frac{1}{2}u_2)^2} du_2 \left(\prod_{j=2}^k dz_j \right) \\ + O_{k,\varepsilon,\mathbf{a}} \left(B^{k-1+12k\varepsilon} \left(TB^{-\frac{1}{2k}} + T^{-1} \right) \right)$$

where $\alpha' = (-\frac{1}{2}u_2, u_2, 0, \dots, 0)$ and $\beta' = (-\frac{1}{2}u_2, 0, \dots, 0)$. Moving c_{u_2} to $c_{u_2} = \frac{1}{2k} - 6k\varepsilon$ picking up the pole at $u_2 = 0$, we then obtain

$$N_{1,\varepsilon}''(B) = C_{1,2}(T) B^{k-1} \log B + C_{1,3}(T) B^{k-1} + O_{k,\varepsilon,\mathbf{a}} \left(B^{k-1+12k\varepsilon} \left(TB^{-\frac{1}{4k}} + T^{-1} \right) \right),$$

where

$$C_{1,2}(T) := \frac{1}{2} \left(\prod_{j=2}^k \frac{1}{2\pi i} \int'_{(c_{z_j})} \right) \frac{Q_{\mathbf{0},\mathbf{0}}(\mathbf{z}, \mathbf{0})}{\zeta(k)^2} \left(\prod_{j=2}^k dz_j \right),$$

$$C_{1,3}(T) := - \left(\prod_{j=2}^k \frac{1}{2\pi i} \int'_{(c_{z_j})} \right) \frac{Q_{\mathbf{0},\mathbf{0}}(\mathbf{z}, \mathbf{0})}{\zeta(k)^2} \left(\gamma + \frac{\partial}{\partial u_2} Q_{\alpha', \beta'}(\mathbf{z}, \mathbf{w}')|_{u_2=0} + \frac{\zeta'(k)}{\zeta(k)} \right) \left(\prod_{j=2}^k dz_j \right)$$

with $\mathbf{z} = (k-1 - \sum_{j=2}^k z_j, z_2, \dots, z_k)$. By (4.3) and the analogous bound for the logarithmic derivative of $Q_{\alpha', \beta'}$ we have

$$C_{1,2}(T) = C'_{1,2} + O(T^{-1}), \quad C_{1,3}(T) = C'_{1,3} + O(T^{-1}),$$

where $C'_{1,2}$ and $C'_{1,3}$ defined as $C_{1,2}, C_{1,3}$ but without the truncation at $|\Im(z_i)| \leq T$ in the integrals. Finally, we write $C'_{1,2}$ as $C'_{1,2} = \mathfrak{S}(\mathbf{a})\sigma_1(\mathbf{a})$, where

$$\mathfrak{S}(\mathbf{a}) := \frac{1}{2} \sum_{\ell \geq 1} \frac{\prod_{i=1}^k (a_i, \ell)}{\ell^k} \frac{\varphi(\ell)}{\zeta(k)^2}$$

and

$$\sigma'_1(\mathbf{a}) := \left(\prod_{i=2}^k \frac{1}{2\pi i} \int_{(c_{z_i})} \right) \frac{2^k \pi^{\frac{k}{2}-1} |a_1|^{k-2-z_2-\dots-z_k} \Gamma(\frac{2-k+z_2+\dots+z_k}{2})}{k-1-z_2-\dots-z_k \Gamma(\frac{k-1-z_2-\dots-z_k}{2})} \prod_{i=2}^k \frac{\Gamma(\frac{1-z_i}{2})}{\Gamma(\frac{z_i}{2})} \frac{dz_i}{|a_i|^{1-z_i} z_i^2}.$$

Summarizing, we proved

$$N_1(B) = B^{k-1} (\mathfrak{S}(\mathbf{a})\sigma'_1(\mathbf{a}) \log T + C'_{1,1} + C'_{1,3}) + O_{k,\varepsilon,\mathbf{a}}(B^{k-1+26k\varepsilon} (TB^{-\frac{1}{4k}} + T^7 B^{-\frac{1}{2k}} + T^{-\frac{1}{2}+\frac{1}{k}})).$$

Thus, Theorem 4 follows by taking $T = B^{\frac{1}{15k-2}}$ and applying the following Lemma.

Lemma 3. *For $k \geq 3$ we have $\sigma'_1(\mathbf{a}) = \sigma_1(\mathbf{a})$ where $\sigma_1(\mathbf{a})$ is as in (1.8).*

Proof. We only give a sketch, leaving the problem of justifying the manipulation of certain conditionally convergent integrals to the interested reader.

First, by symmetry, we observe that

$$\sigma_1(\mathbf{a}) = 2^k \sum_{\epsilon = (\pm 2^1, \dots, \pm k^1) \in \{\pm 1\}^{k-1}} \left(\prod_{i=2}^k \int_0^1 \int_0^1 \right) \chi_{(0,1)} \left(\sum_{i=2}^k (\pm_i a_i x_i y_i / a_1) \right) \prod_{i=2}^k dx_i dy_i.$$

Then, we detect the characteristic function $\chi_{(0,1)}$ using its Mellin transform obtaining

$$\sigma_1(\mathbf{a}) = \sum_{\epsilon=(\pm_2 1, \dots, \pm_k 1) \in \{\pm 1\}^{k-1}} \left(\prod_{i=2}^k \int_0^1 \int_0^1 \right) \frac{1}{2\pi i} \int_{(c_{z_1})} \frac{2^k (\sum_{i=2}^k (\pm_i a_i x_i y_i / a_1))^{z_1-1}}{1-z_1} \\ \times \chi_{\mathbb{R}_{>0}} \left(\sum_{i=2}^k (\pm_i a_i x_i y_i / a_1) \right) dz_1 \prod_{i=2}^k dx_i dy_i$$

with $c_{z_1} = \frac{1}{k}$. Then, we use Lemma 8 below with $B = 0^3$ obtaining

$$\sigma_1(\mathbf{a}) = \sum_{\epsilon=(\pm_2 1, \dots, \pm_k 1) \in \{\pm 1\}^{k-1}} \left(\prod_{i=2}^k \int_0^1 \int_0^1 \right) \left(\prod_{i=1}^k \frac{1}{2\pi i} \int_{(c_{z_i})} - \prod_{i=1}^k \frac{1}{2\pi i} \int_{(c'_{z_i})} \right) \frac{2^k}{1-z_1} \\ \times \frac{G(1-z_1-z_2-\dots-z_k) \Gamma(z_1) \prod_{i=2}^k \Gamma(z_i) |a_i x_i y_i / a_1|^{-z_i}}{1-z_1-z_2-\dots-z_k \Gamma(\sum_{1 \leq i \leq k}^+ z_i) \Gamma(\sum_{1 \leq i \leq k}^- z_i)} \left(\prod_{i=1}^k dz_i \right) \prod_{i=2}^k dx_i dy_i,$$

where $G(s)$ is entire with $G(0) = 1$, \sum_i^\pm indicates the sum is restricted to indexes such that $\pm_i \text{sign}(a_i) = \pm 1$, with $\pm_1 1 := -\text{sign}(a_1)$, and the lines of integrations are $c'_{z_i} = \frac{1}{k-1}$ and $c_{z_i} = \frac{1}{k+1}$ for $i = 2, \dots, k$ and $c_{z_1} = c'_{z_1} = \frac{1}{k}$. Then, we notice that we can take instead $c_{z_i} = c'_{z_i} = \frac{1}{k}$ for $i = 2, \dots, k$ and $c_{z_1} = \frac{1}{k+1}$, $c'_{z_1} = \frac{1}{k-1}$. We take the integral over x_i and y_i inside and execute them, obtaining

$$\sigma_1(\mathbf{a}) = \sum_{\epsilon=(\pm_2 1, \dots, \pm_k 1) \in \{\pm 1\}^{k-1}} \left(\prod_{i=2}^k \frac{1}{2\pi i} \int_{(c_{z_i})} \right) \left(\frac{1}{2\pi i} \int_{(c_{z_1})} - \frac{1}{2\pi i} \int_{(c'_{z_1})} \right) \frac{2^k}{1-z_1} \\ \times \frac{G(1-z_1-z_2-\dots-z_k) \Gamma(z_1) \prod_{i=2}^k \Gamma(z_i) |a_i / a_1|^{-z_i}}{1-z_1-z_2-\dots-z_k \Gamma(\sum_{1 \leq i \leq k}^+ z_i) \Gamma(\sum_{1 \leq i \leq k}^- z_i)} dz_1 \left(\prod_{i=1}^k \frac{dz_i}{(1-z_i)^2} \right).$$

By the residue theorem the difference of the integrals in z_1 is equal to minus the residue at $z_1 = 1 - z_2 - \dots - z_k$ and so

$$\sigma_1(\mathbf{a}) = \sum_{\epsilon=(\pm_2 1, \dots, \pm_k 1) \in \{\pm 1\}^{k-1}} \left(\prod_{i=2}^k \frac{1}{2\pi i} \int_{(c_{z_i})} \right) \frac{2^k}{1-z_1} \frac{\Gamma(z_1) \prod_{i=2}^k \Gamma(z_i) |a_i / a_1|^{-z_i}}{\Gamma(\sum_{1 \leq i \leq k}^+ z_i) \Gamma(\sum_{1 \leq i \leq k}^- z_i)} \prod_{i=2}^k \frac{dz_i}{(1-z_i)^2}$$

where $z_1 := 1 - z_2 - \dots - z_k$. We take the sum over ϵ inside, and evaluate it using (2.8) (notice that since the value of $\pm_1 1$ is fixed we have to multiply by $\frac{1}{2}$). We obtain

$$\sigma_1(\mathbf{a}) = \left(\prod_{i=2}^k \frac{1}{2\pi i} \int_{(c_{z_i})} \right) \frac{2^k \pi^{\frac{k}{2}-1}}{z_2 + \dots + z_k} \frac{\Gamma(\frac{1-z_2-\dots-z_k}{2})}{\Gamma(\frac{z_2+\dots+z_k}{2})} \prod_{i=2}^k \frac{\Gamma(\frac{z_i}{2}) |a_i / a_1|^{-z_i}}{\Gamma(\frac{1-z_i}{2})} \frac{dz_i}{(1-z_i)^2}.$$

³The Lemma is stated for B large to avoid issues coming with the conditional convergence of the integrals. To make this rigorous it's enough to take a larger B , and then later in the argument recompute the sum over ν using (7.22).

Making the change of variables $z_i \rightarrow 1 - z_i$ for all $i = 2, \dots, k$ we obtain $\sigma'_1(\mathbf{a})$. \square

5 The region of absolute convergence

In this section we prove a bound for $\mathcal{A}_{\mathbf{a};\alpha,\beta}(s)$ in the region of absolute convergence. We remark that if we were not concerned with the uniformity in k , then an easier argument would have sufficed.

Lemma 4. *Let $k \geq 3$, $\mathbf{a} \in \mathbb{Z}_{\neq 0}^k$ and $\alpha, \beta \in \mathbb{C}^k$ with $|\Re(\alpha_i)|, |\Re(\beta_i)| \leq \frac{1}{2(k-1)}$ for all $i = 1, \dots, k$. Then $\mathcal{A}_{\mathbf{a};\alpha,\beta}(s)$ converges absolutely on $\Re(s) > 1 - \frac{1}{k} - \frac{1}{k} \sum_{i=1}^k \min(\Re(\alpha_i), \Re(\beta_i))$. Moreover, if*

$$\Re(s) \geq 1 - \frac{1}{k} + \frac{\varepsilon}{k} - \frac{1}{k} \sum_{i=1}^k \min(\Re(\alpha_i), \Re(\beta_i)) \quad (5.1)$$

for some $\varepsilon > 0$, then $\mathcal{A}_{\mathbf{a};\alpha,\beta}(s) \ll_{\varepsilon} \left(\frac{Ak}{\varepsilon}\right)^{4k}$ where the implicit constant depends on ε only and A is an absolute constant.

Proof. Clearly, we can assume $s, \alpha_i, \beta_i \in \mathbb{R}$ and $\alpha_i \leq \beta_i$ for all $i = 1, \dots, k$ and that $\varepsilon < \frac{1}{8}$. Also, it is sufficient to establish the claimed bound for $s = 1 - \frac{1}{k} - \frac{1}{k} \sum_{i=1}^k \alpha_i + \frac{4\varepsilon}{k}$.

We have $\tau_{\alpha,\beta}(n) \leq d(n)n^{\max(-\alpha,-\beta)}$ and so

$$\mathcal{A}_{\mathbf{a};\alpha,\beta}(s) \leq \sum_{i=1}^k \sum_{\substack{a_1 n_1 + \dots + a_k n_k = 0, \\ \max_{j=1}^k (n_j) = n_i}} \frac{d(n_1) \cdots d(n_k)}{n_1^{s_1} \cdots n_k^{s_k}} = \sum_{i=1}^k S_i,$$

say, where we wrote $s_i := s + \alpha_i$. By hypothesis $s_1 - \varepsilon > 1 - \frac{1}{k} - \frac{1}{k-1} - \varepsilon > 0$ and so

$$S_1 \ll_{\varepsilon} \sum_{\substack{a_1 n_1 + \dots + a_k n_k = 0, \\ \max_{j=1}^k (n_j) = n_1}} \frac{d(n_2) \cdots d(n_k)}{n_1^{s_1 - \varepsilon} n_2^{s_2} \cdots n_k^{s_k}} \leq \sum_{\ell=1}^{k-1} \sum_{\substack{n_2, \dots, n_k \geq 1, \\ \max_{r=2}^k (n_r) = n_{\ell}}} \frac{d(n_2) \cdots d(n_k)}{n_{\ell}^{s_1 - \varepsilon} n_2^{s_2} \cdots n_k^{s_k}} = \sum_{\ell=2}^k Z_{1,\ell},$$

say. We write

$$c_r := 1 - s_r + \xi_r \frac{\varepsilon}{k} = \frac{1}{k} + \frac{1}{k} \sum_{\substack{i=1, \\ i \neq r}}^k \alpha_i + \frac{k-1}{k} \alpha_i - \varepsilon + \xi_r \frac{\varepsilon}{k}$$

where ξ_r is any real number in the interval $(1, 2)$ such that $|c_r| > \frac{\varepsilon}{2k}$ (notice that since $|\alpha_r| \leq \frac{1}{2(k-1)}$ we also have $c_r > -\varepsilon + \xi_r \frac{\varepsilon}{k} > -\frac{1}{8}$). Now,

$$\chi_{[0,1]}(x) \leq 2 \left(1 - \frac{x}{2}\right) \chi_{[0,2]}(x) = \frac{1}{\pi i} \int_{(1)} (x/2)^{-w} \frac{dw}{w(w+1)} = 2\chi_I(r) + \frac{1}{\pi i} \int_{(c_r)} (x/2)^{-w} \frac{dw}{w(w+1)},$$

by the residue theorem, where I is the set of $r \in \{3, \dots, k\}$ for which $c_r < 0$ and χ_I is the characteristic function of the set I . We replace the condition $n_r \leq n_2$ for all $r = 3, \dots, k$ by this formula and obtain

$$\begin{aligned} Z_{1,2} &\leq \sum_{n_2, \dots, n_k \geq 1} \frac{d(n_2) \cdots d(n_k)}{n_2^{s_1+s_2-\varepsilon} n_3^{s_3} \cdots n_k^{s_k}} \prod_{r=3}^k \left(2\chi_I(r) + \frac{1}{\pi i} \int_{(c_r)} \left(\frac{n_r}{2n_2} \right)^{-w_r} \frac{dw_r}{w_r(w_r+1)} \right) \\ &= \sum_{J \subseteq I} 2^{|J|} \prod_{j \in J} \zeta\left(1 + \frac{\varepsilon}{k}\right)^2 \sum_{n_2 \geq 1} \frac{d(n_2)}{n_2^{s_1+s_2-\varepsilon}} \prod_{\ell \in L} \sum_{n_\ell \geq 1} \frac{d(n_\ell)}{n_\ell^{s_\ell}} \frac{1}{\pi i} \int_{(c_\ell)} \left(\frac{n_\ell}{2n_2} \right)^{-w_\ell} \frac{dw_\ell}{w_\ell(w_\ell+1)}, \end{aligned}$$

since $s_i > 1 + \frac{\varepsilon}{k}$ for all $i \in I$, and where $L := \{3, \dots, k\} \setminus J$. (Here and below the exchanges in the orders of sums and integrals are justified by the absolute convergence). Exchanging the order of summation and integration and summing the Dirichlet series we obtain

$$Z_{1,2} \leq \sum_{J \subseteq I} 2^{|J|} \left(\prod_{j \in J} \zeta(s_j) \right) \left(\prod_{\ell \in L} \frac{1}{\pi i} \int_{(c_\ell)} \zeta(s_\ell + w_\ell)^2 \right) \zeta\left(s_1 + s_2 - \varepsilon - \sum_{\ell \in L} w_\ell\right)^2 \prod_{\ell \in L} \frac{dw_\ell}{w_\ell(w_\ell+1)}.$$

The real part of the argument of the last ζ in the above equation is

$$\begin{aligned} s_1 + s_2 + \sum_{\ell \in L} \left(s_\ell - \xi_\ell \frac{\varepsilon}{k} \right) - |L| - \varepsilon &\geq (|L| + 2)s - |L| - \varepsilon - 2\frac{\varepsilon|L|}{k} + \sum_{\ell \in L \cup \{1,2\}} \alpha_\ell \\ &= 1 + \frac{k - |L| - 2}{k} + \frac{2|L| - k + 8}{k} \varepsilon - \frac{|L| + 2}{k} \sum_{j \in J} \alpha_j + \frac{k - |L| - 2}{k} \sum_{\ell \in L \cup \{1,2\}} \alpha_\ell \end{aligned}$$

by the definition of s_ℓ and since $\xi_\ell \leq 2$ and $s = 1 - \frac{1}{k} - \frac{1}{k} \sum_{i=1}^k \alpha_i + \frac{4\varepsilon}{k}$. Since $|\alpha_i| \leq \frac{1}{2(k-1)}$ for all $i = 1, \dots, k$, then the above expression is

$$\begin{aligned} &\geq 1 + \frac{k - |L| - 2}{k} + \frac{2|L| - k + 8}{k} \varepsilon - \frac{(k - |L| - 2)(|L| + 2)}{k(k-1)} \\ &= 1 + \frac{(k - |L| - 2)(k - |L| - 3)}{k(k-1)} + \frac{2|L| - k + 8}{k} \varepsilon = 1 + \frac{m(m-1)}{k(k-1)} + \frac{k - 2m + 4}{k} \varepsilon, \end{aligned}$$

where $m := k - 2 - |L|$ (so that $0 \leq m \leq k - 2$). Thus, if $m < k/2$ then $\Re(s_1 + s_2 - \varepsilon - \sum_{\ell \in L} w_\ell) \geq 1 + \frac{4}{k} \varepsilon$, and the same holds if $m \geq k/2$ since in this case we have

$$1 + \frac{m(m-1)}{k(k-1)} + \frac{k - 2m + 4}{k} \varepsilon \geq 1 + \frac{(k-2)}{4(k-1)} - \varepsilon + \frac{8}{k} \varepsilon \geq 1 + \frac{1}{8} - \varepsilon + \frac{8}{k} \varepsilon > 1 + \frac{8}{k} \varepsilon$$

for $k \geq 3$ and $\varepsilon < \frac{1}{8}$. It follows that

$$Z_{1,2} \leq \sum_{J \subseteq I} \frac{2^{|J|}}{\pi^{|L|}} \zeta\left(1 + 4\frac{\varepsilon}{k}\right)^2 \left(\prod_{j \in J} \zeta\left(1 + \frac{\varepsilon}{k}\right)^2 \right) \prod_{\ell \in L} \zeta\left(1 + \xi_\ell \frac{\varepsilon}{k}\right)^2 \int_{(c_\ell)} \frac{|dw_\ell|}{|w_\ell(w_\ell+1)|}$$

since by our choice we have $s_\ell + c_\ell = 1 + \xi_\ell \frac{\varepsilon}{k} > 1$ for all $\ell \in L$. Finally, since $|c_\ell| > \frac{\varepsilon}{2k}$ and $c_\ell > -\frac{1}{2}$ we have

$$\int_{(c_\ell)} \frac{|dw_\ell|}{|w_\ell(w_\ell + 1)|} \leq \frac{2k}{\varepsilon} \int_{\mathbb{R}} \frac{1}{|(1+ix)(\frac{1}{2}+ix)|} \leq \frac{2k}{\varepsilon} \int_{\mathbb{R}} \frac{1}{|\frac{1}{2}+ix|^2} = \frac{4\pi k}{\varepsilon}$$

and so, since $s_r > 1 + \xi_r \frac{\varepsilon}{k}$ for all $r \in I$ and $s_\ell + c_\ell > \frac{\varepsilon}{k}$ for all $r = 3, \dots, k$, we have

$$Z_{1,2} \leq \zeta(1 + \frac{\varepsilon}{k})^{2k-2} \sum_{J \subseteq I} \frac{2^{|J|}}{\pi^{|L|}} \left(\frac{4\pi k}{\varepsilon} \right)^{|L|} \leq k! \left(\frac{8k}{\varepsilon} \right)^k \zeta(1 + \frac{\varepsilon}{k})^{2k-2} \ll \left(\frac{Ak}{\varepsilon} \right)^{4k}$$

for some fixed $A > 0$. The same bound clearly holds for $Z_{i,j}$ for all i, j and so the Lemma follows. \square

6 Proof of Theorem 3

Throughout the rest of the paper, we shall write

$$X_{s,k,\mathbf{a},\varepsilon} := \frac{k}{\varepsilon} (1 + |s|) \max_{i=1}^k |a_i|, \quad Y_{\alpha,\beta} := (1 + \max_{i=1}^k (|\Im(\alpha_i)| + |\Im(\beta_i)|)). \quad (6.1)$$

Then, we notice that instead of (2.6), it is enough to prove

$$\mathcal{A}_{\mathbf{a};\alpha,\beta}(s) - \mathcal{M}_{\mathbf{a};\alpha,\beta}(s) \ll_{\varepsilon} (X_{s,k,\mathbf{a},\varepsilon} Y_{\alpha,\beta})^{Ak} \quad (6.2)$$

for $\Re(s) \geq 1 - \frac{2-2\eta_{\alpha,\beta}-\varepsilon}{k+1}$. Indeed, we can apply the Phragmen-Lindelöf principle on the region $1 - \frac{2-2\eta_{\alpha,\beta}-\varepsilon}{k+1} \leq \Re(s) \leq 1 - \frac{1-\varepsilon}{k} + \frac{\eta_{\alpha,\beta}}{k+1}$ using (6.2) on the left boundary line and the bound for $\mathcal{A}_{\mathbf{a};\alpha,\beta}(s)$ given in Lemma 4 with a trivial bound for $\mathcal{M}_{\mathbf{a};\alpha,\beta}(s)$ on the line $\Re(s) = 1 - \frac{1-\varepsilon}{k} + \frac{\eta_{\alpha,\beta}}{k+1}$. Also, in (2.6) we can take $(1 + |s|)^7$ rather than $(1 + |s|)^A$ since for $s = \sigma + it$ one has $\mathcal{A}_{\mathbf{a};\alpha,\beta}(s) = \mathcal{A}_{\mathbf{a};\alpha+\mathbf{t}',\beta+\mathbf{t}'}(\sigma)$ where $\mathbf{t}' := (t, \dots, t)$.

Furthermore, we notice it is sufficient to prove Theorem 3 in the case $|\alpha_i - \beta_i| > \frac{\varepsilon}{k}$ for $i = 1, \dots, k$. Indeed, assume we have proved Theorem 3 in this restricted case and let

$$\mathcal{F}_{\mathbf{a};\alpha,\beta}(s) := \frac{1}{2\pi i} \int_{|\xi_1|=4\frac{\varepsilon}{k}} (\mathcal{A}_{\mathbf{a};\alpha'(\xi),\beta}(s) - \mathcal{M}_{\mathbf{a};\alpha'(\xi),\beta}(s)) \frac{d\xi}{\xi},$$

where $\alpha'(\xi) = (\alpha'_1, \dots, \alpha'_k) := (\alpha_1 + \xi, \alpha_2, \dots, \alpha_k)$. Then, $\mathcal{F}_{\mathbf{a};\alpha,\beta}(s)$ defines a holomorphic function in (s, α, β) in the domain

$$D := \left\{ (s, \alpha, \beta) \mid \begin{array}{l} |\alpha_1 - \beta_1| < \frac{2\varepsilon}{k}, |\alpha_i - \beta_i| > \frac{\varepsilon}{k}, \forall i \geq 2, |\Re(\alpha_i)|, |\Re(\beta_i)| \leq \frac{1-9\varepsilon}{2(k-1)} \quad \forall i \geq 1, \\ \Re(s) > 1 - \frac{2-2\eta_{\alpha,\beta}-10\varepsilon}{k+1} \end{array} \right\}.$$

Moreover, by Cauchy's theorem we have $\mathcal{F}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s) = \mathcal{A}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s) - \mathcal{M}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ in the subset of D with $\Re(s) > 4$, since $\mathcal{A}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ is clearly holomorphic also for $\alpha_1 = \beta_1$ when the series defining it is absolutely convergent and the same holds for $\mathcal{M}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ since the poles of $\mathcal{M}_{\mathbf{a};\boldsymbol{\alpha}'(\xi),\boldsymbol{\beta}}(s)$ at $\alpha_1 = \beta_1$ cancel. Thus, we have obtained the analytic continuation of $\mathcal{A}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s) - \mathcal{M}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ on the domain given by D without any condition on $|\alpha_1 - \beta_1|$ and by the above integral representation the bound (6.2) holds also in the case $|\alpha_1 - \beta_1| \leq \frac{\varepsilon}{k}$. Repeating this procedure for $i = 2, \dots, k$ one obtains Theorem 3 in the general case. Thus, in the following we will assume $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega_k$ with

$$\Omega_k := \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{C}^k \times \mathbb{C}^k \mid \frac{\varepsilon}{k} < |\alpha_i - \beta_i|, \quad |\alpha_i|, |\beta_i| < \frac{1}{2(k-1)} \forall i = 1, \dots, k\}. \quad (6.3)$$

Finally, we can and shall assume $\sigma := \Re(s) \ll 1$.

We localize the variables of summation by introducing partitions of unity

$$\sum_N^\dagger P(x/N) = x, \quad \forall x > 0,$$

such that $\sum_{X^{-1} \leq N \leq X}^\dagger 1 \ll \log X$ and with $P(x)$ supported on $1 \leq x \leq 2$ and satisfying $P^{(j)}(x) \ll j^{Aj}$ for some $A > 0$. Notice that under these conditions, the Mellin transform of $P(x)$,

$$\tilde{P}(s) := \int_0^\infty P(x)x^{s-1}dx,$$

is entire and satisfies

$$\tilde{P}(\sigma + it) \ll (1 + r + |\sigma|)^{Ar} C^{|\sigma|} (1 + |t|)^{-r}, \quad \forall r \geq 0 \quad (6.4)$$

for some $C > 0$. Thus, for s satisfying (5.1) we can write

$$\mathcal{A}_{\mathbf{a};\boldsymbol{\alpha},\boldsymbol{\beta}}(s) := \sum_{N_1, \dots, N_k}^\dagger A_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s),$$

where

$$A_{\boldsymbol{\alpha},\boldsymbol{\beta}}(s) := \sum_{a_1 n_1 + \dots + a_k n_k = 0} \frac{\sigma_{\alpha_1 - \beta_1}(n_1) \cdots \sigma_{\alpha_k - \beta_k}(n_k)}{n_1^{s+\alpha_1} \cdots n_k^{s+\alpha_k}} P(n_1/N_1) \cdots P(n_k/N_k) \quad (6.5)$$

and, here and in the following, we omit to indicate the dependence on \mathbf{a} and N_1, \dots, N_k to save notation. The main step in the proof consists in the following lemma which we shall prove in Section 7.

Lemma 5. *We have*

$$A_{\alpha,\beta}(s) = \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, \\ |\mathcal{I}| > \frac{k+1}{2}}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i, \beta_i\} \quad \forall i \in \mathcal{I}, \\ (\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j) \quad \forall j \in \mathcal{J}}} \prod_{i \in \mathcal{I}} \zeta(1 - \alpha_i + \beta_i) Z_{\mathcal{I}; \alpha^*, \beta^*}(s) + E_1(s), \quad (6.6)$$

where $Z_{\mathcal{I}; \alpha, \beta}$ is as in (7.24) and $E_1(s)$ is holomorphic in $(s, \alpha, \beta) \in \mathbb{C} \times \Omega_k$ and satisfies

$$E_1(s) \ll_{\varepsilon} X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_{\max}^{\eta_{\alpha,\beta} + 2\varepsilon} \times \left(\frac{N_{\max}^{\frac{k-1}{2}}}{(N_1 \cdots N_k)^{\sigma}} + \frac{N_{\max}^{\frac{k-1}{2} - \frac{1}{k-1}}}{(N_1 \cdots N_k)^{\sigma - \frac{1}{k-1}}} + \frac{(N_1 \cdots N_k)^{1 - \frac{1}{k} - \sigma}}{N_{\max}} + N_{\max}^{\frac{k-3}{2} - \frac{k+1}{2}\sigma} \right) \quad (6.7)$$

for $\frac{1}{2} \leq \sigma \leq 1$, $(\alpha, \beta) \in \Omega_k$, with $\eta_{\alpha,\beta}$ is as in (2.1) and $N_{\max} := \max_{i=1}^k N_i$.

In Section 8 we shall prove the following lemma.

Lemma 6. *We have*

$$A_{\alpha,\beta}(s) = \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, \\ |\mathcal{I}| > \frac{k+1}{2}}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i, \beta_i\} \quad \forall i \in \mathcal{I}, \\ (\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j) \quad \forall j \in \mathcal{J}}} \prod_{i \in \mathcal{I}} \zeta(1 - \alpha_i + \beta_i) Z_{\mathcal{I}; \alpha^*, \beta^*}(s) + E_2(s), \quad (6.8)$$

where $E_2(s)$ is holomorphic in $(s, \alpha, \beta) \in \mathbb{C} \times \Omega_k$ and satisfies

$$E_2(s) \ll_{\varepsilon} X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^k (N_1 \cdots N_k)^{1-\sigma} N_{\max}^{-1+\eta_{\alpha,\beta}+\varepsilon}. \quad (6.9)$$

Finally, in Section 9 we complete the proof of Theorem 3 summing over the partitions of unity and computing the minimum of the two error terms.

7 Proof of Lemma 5

Since both $A_{\alpha,\beta}(s)$ and the main term on the right side of (6.6) are symmetric in N_1, \dots, N_k and with respect to the change $\mathbf{a} \leftrightarrow -\mathbf{a}$ (cf. Remark 9 at the end of Subsection 7.5), we can and shall assume that N_1 is the maximum among the N_i and that $a_1 < 0$.

7.1 Separating the variables

The condition $-a_1 n_1 = a_2 n_2 + \cdots + a_k n_k$ can be used to eliminate the variable n_1 in the definition (6.5) of $A_{\alpha,\beta}(s)$, by adding the conditions

$$a_2 n_2 + \cdots + a_k n_k \equiv 0 \pmod{|a_1|}, \quad a_2 n_2 + \cdots + a_k n_k > 0$$

and replacing each occurrences of n_1 by $(a_2 n_2 + \cdots + a_k n_k)/|a_1|$. This poses the problem of expressing $\tau_{\alpha_1, \beta_1}(n_1) = n_1^{-\alpha} \sigma_{\alpha-\beta}(n_1)$ in a more flexible way, which we achieve by the following modification of Ramanujan's identity (1.10) which also allows us to remove the above congruence condition.

Lemma 7. *Let $a, m \in \mathbb{N}$ and $\gamma \in \mathbb{C}$. Then we have*

$$\delta_{a|n} \sigma_\gamma(n/a) = \frac{1}{a} \sum_{\ell} \frac{c_\ell(n)}{\ell^{1-\gamma}} (\ell, a)^{1-\gamma} v_\gamma \left(\frac{a\ell^2}{(a, \ell)^2 n} \right) + n^\gamma \sum_{\ell} \frac{c_\ell(n)}{\ell^{1+\gamma}} (\ell, a)^{1+\gamma} v_{-\gamma} \left(\frac{a\ell^2}{(a, \ell)^2 n} \right),$$

where $\delta_{a|n} = 1$ if $a|n$ and $\delta_{a|n} = 0$ otherwise, $c_\ell(n) := \sum_{h \pmod{n}}^* e\left(\frac{h\ell}{n}\right)$ is the Ramanujan sum and for any $c_w > |\Re(\gamma)|$

$$v_\gamma(x) = \int_{(c_w)} x^{-\frac{w}{2}} \zeta(1-\gamma+w) \frac{G(w)}{w} dw, \quad (7.1)$$

where $G(w)$ is any even entire function which decays faster than any polynomial in vertical strips and is such that $G(0) = 1$.

Proof. We start by observing that for $\Re(s) > 1$ we have

$$\delta_{a|m} \sum_{\ell \geq 1} \frac{c_\ell(m/a)}{\ell^s} = \frac{1}{a} \sum_{\ell \geq 1} \frac{c_\ell(m)}{\ell^s} (\ell, a)^s. \quad (7.2)$$

Indeed, by the orthogonality of additive characters, the left hand side is

$$\begin{aligned} \frac{1}{a} \sum_{g=1}^a \sum_{\ell \geq 1} \frac{1}{\ell^s} \sum_{\substack{h=1, \\ (h, \ell)=1}}^{\ell} e\left(\frac{mh + mg\ell}{a\ell}\right) &= \frac{1}{a} \sum_{\ell \geq 1} \frac{1}{\ell^s} \sum_{\substack{h \pmod{a\ell}, \\ (h, \ell)=1}} e\left(\frac{mh}{a\ell}\right) \\ &= \frac{1}{a} \sum_{\ell \geq 1} \frac{1}{\ell^s} \sum_{\substack{d|a, \\ (a/d, \ell)=1}} \sum_{\substack{h \pmod{d\ell}, \\ (h, d\ell)=1}} e\left(\frac{mh}{d\ell}\right) = \frac{1}{a} \sum_{\ell \geq 1} \frac{1}{\ell^s} \sum_{\substack{d|a, \\ (a, d\ell)=d}} c_{d\ell}(m) = \frac{1}{a} \sum_{\substack{\ell' \geq 1, \\ (a, \ell')=d}} \frac{1}{(\ell'/d)^s} c_{\ell'}(m), \end{aligned}$$

as claimed. By Ramanujan's identity (1.10), one has that (7.2) gives

$$\delta_{a|m} \sigma_{1-s}(m/a) = \zeta(s) \frac{1}{a} \sum_{\ell \geq 1} \frac{c_\ell(m)}{\ell^s} (\ell, a)^s \quad (7.3)$$

for $\Re(s) > 1$. Now, by the residue theorem for any $c_w > |\Re(\gamma)|$ we have

$$\begin{aligned} \sigma_\gamma(m/a) &= \int_{(c_w)} (m/a)^{\frac{w}{2}} \sigma_{\gamma-w}(m/a) \frac{G(w)}{w} dw - \int_{(-c_w)} (m/a)^{\frac{w}{2}} \sigma_{\gamma-w}(m/a) \frac{G(w)}{w} dw \\ &= \int_{(c_w)} (m/a)^{\frac{w}{2}} \sigma_{\gamma-w}(m/a) \frac{G(w)}{w} dw + \int_{(c_w)} (m/a)^{\gamma+\frac{w}{2}} \tau_{-\gamma-w}(m/a) \frac{G(w)}{w} dw \end{aligned}$$

since $\sigma_{\gamma+w}(m) = m^{\gamma+w}\tau_{-\gamma-w}(m)$ and $G(-w) = G(w)$, and so the lemma then follows by (7.3). \square

Remark 5. *It will be convenient to take $G(w)$ with a zero which cancels the pole of the zeta-function in the definition (7.1) of v . Thus we take*

$$G_{\alpha_1, \beta_1}(w) := \frac{w^2 - (\alpha_1 - \beta_1)^2 \xi(\frac{1}{2} + w)}{(\alpha_1 - \beta_1)^2 \xi(\frac{1}{2})},$$

where $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\pi/2}\Gamma(\frac{1}{2}w)\zeta(w)$ is the Riemann ξ -function. Notice that $G(0) = 1$ and that by the functional equation we have $G_{\alpha_1, \beta_1}(w) = G_{\alpha_1, \beta_1}(-w)$. Also, by Stirling's formula we have

$$\begin{aligned} G_{\alpha_1, \beta_1}(w) &\ll (k/\varepsilon)^2 e^{-\frac{\pi}{8}|\Im(w)|} (1 + |\Re(w)|)^{A(|\Re(w)|+1)} \\ &\ll (k/\varepsilon)^2 (1+r + |\Re(w)|)^{A(1+r+|\Re(w)|)} (1+|w|)^{-r} \end{aligned} \quad (7.4)$$

for any $r \geq 0$.

Applying this lemma we obtain

$$A_{\alpha, \beta}(s) = \sum_{\substack{\{\alpha_1^*, \beta_1^*\} = \{\alpha_1, \beta_1\}, \\ (\alpha_i^*, \beta_i^*) = (\alpha_i, \beta_i) \quad \forall i=2, \dots, k}} K_{\alpha^*, \beta^*}(s),$$

where the sum is over $\alpha^* = (\alpha_1^*, \dots, \alpha_k^*)$, $\beta^* = (\beta_1^*, \dots, \beta_k^*)$ and

$$\begin{aligned} K_{\alpha, \beta} &:= \frac{1}{|a_1|} \sum_{\substack{n_2, \dots, n_k \geq 1, \\ n_1 = (a_2 n_2 + \dots + a_k n_k) / |a_1| > 0}} n_1^{-\alpha_1} \sum_{\ell \geq 1} \frac{c_\ell (a_2 n_2 + \dots + a_k n_k)}{\ell^{1-\alpha_1+\beta_1}} (\ell, a_1)^{1-\alpha_1+\beta_1} \times \\ &\times \frac{\tau_{\alpha_2, \beta_2}(n_2) \cdots \tau_{\alpha_k, \beta_k}(n_k)}{(n_1 \cdots n_k)^s} P(n_1/N_1) \cdots P(n_k/N_k) v_{\alpha_1-\beta_1} \left(\frac{\ell^2}{(\ell, a_1)^2 n_1} \right). \end{aligned}$$

Next, we express P and v using their Mellin transforms so that, after making the change of variables $u_i \rightarrow u_i - s$ for all $i \in \{1, \dots, k\}$, we obtain

$$\begin{aligned} K_{\alpha, \beta} &= \sum_{\pm_2 a_2 n_2 \pm_3 \cdots \pm_k a_k n_k > 0} \sum_{\ell \geq 1} \sum_{h \pmod{\ell}}^* \frac{1}{(2\pi i)^{k+1}} \int_{(c_w, c_{u_1}, \dots, c_{u_k})} \frac{N_1^{u_1-s} \cdots N_k^{u_k-s}}{(\ell/(a_1, \ell))^{1-\alpha_1+\beta_1+w}} \\ &\times \frac{\tau_{\alpha_2, \beta_2}(n_2) \cdots \tau_{\alpha_k, \beta_k}(n_k) c_\ell (a_2 n_2 + \dots + a_k n_k) (\ell, a_1)^{1-\alpha_1+\beta_1+w}}{(a_2 n_2 + \dots + a_k n_k)^{\alpha_1+u_1-\frac{w}{2}} n_2^{u_2} \cdots n_k^{u_k} |a_1|^{1-\alpha_1-u_1+\frac{w}{2}}} \\ &\times \tilde{P}(u_1-s) \cdots \tilde{P}(u_k-s) \zeta(1-\alpha_1+\beta_1+w) \frac{G_{\alpha_1, \beta_1}(w)}{w} dw du_1 \cdots du_k, \end{aligned} \quad (7.5)$$

with lines of integrations

$$c_w = |\Re(\alpha_1 - \beta_1)| + 2\varepsilon, \quad c_{u_1} = -4k - \Re(\alpha_1) + \frac{c_w}{2} + \varepsilon, \quad c_{u_2} = \dots = c_{u_k} = 5k. \quad (7.6)$$

Next, we separate the variables in the expression $(a_2 n_2 + \dots + a_k n_k)^{\frac{1}{2} + \alpha_1 + u_1 - \frac{w}{2}}$ using the following lemma which we quote from Section 10 of [Bet] in a slightly adapted form.

Lemma 8. *Let $\kappa \geq 2$ and $x_1, \dots, x_\kappa > 0$. Let $\epsilon = (\pm_1, \dots, \pm_\kappa 1) \in \{\pm 1\}^\kappa$, with $\pm_1 1 = -1$. Let $B \in \mathbb{Z}_{\geq 0}$ be such that $\frac{\kappa}{2} + \frac{1}{2} < \Re(v_1) < B + 1$. Moreover, let $c_{v_2}, \dots, c_{v_\kappa}, c'_{v_2}, \dots, c'_{v_\kappa} > 0$ be such that*

$$\Re(v_1) + c_{v_2} + \dots + c_{v_\kappa} < B + 1 < \Re(v_1) + c'_{v_2} + \dots + c'_{v_\kappa}.$$

Then

$$\begin{aligned} & (\pm_2 x_2 \pm_3 \dots \pm_\kappa x_\kappa)^{v_1 - 1} \chi_{\mathbb{R}_{>0}}(\pm_2 x_2 \pm_3 \dots \pm_\kappa x_\kappa) \\ &= \sum_{\substack{\nu = (\nu_2, \dots, \nu_\kappa) \in \mathbb{Z}_{\geq 0}^{\kappa-1}, \\ \nu_2 + \dots + \nu_\kappa = B, \\ \nu_i = 0 \text{ if } \pm_i = -1}} \frac{B!}{\nu_2! \dots \nu_\kappa!} \frac{1}{(2\pi i)^{\kappa-1}} \left(\int_{(c_{v_2}, \dots, c_{v_\kappa})} - \int_{(c'_{v_2}, \dots, c'_{v_\kappa})} \right) \frac{\Psi_{\epsilon, B}(v_1, \dots, v_\kappa)}{x_2^{v_2 - \nu_2} \dots x_\kappa^{v_\kappa - \nu_\kappa}} dv_2 \dots dv_\kappa, \end{aligned}$$

where

$$\begin{aligned} \Psi_{\epsilon, B}(s_1, \dots, s_\kappa) &:= \frac{\Gamma(s_1) \dots \Gamma(s_\kappa)}{\Gamma(V_\epsilon^+(s_1, \dots, s_\kappa)) \Gamma(V_\epsilon^-(s_1, \dots, s_\kappa))} \frac{G(B + 1 - s_1 - \dots - s_\kappa)}{B + 1 - s_1 - \dots - s_\kappa}, \\ V_\epsilon^\pm(s_1, \dots, s_\kappa) &:= \sum_{\substack{1 \leq i \leq \kappa, \\ \pm_i 1 = \pm 1}} s_i \end{aligned}$$

and $G(s)$ is any entire function such that $G(0) = 1$ and $G(\sigma + it) \ll e^{-C_1|t|} (1 + |\sigma|)^{C_2|\sigma|}$ for some fixed $C_1, C_2 > 0$. Moreover, writing $s_i = \sigma_i + it_i$ for $i = 1, \dots, \kappa$, we have

$$\Psi_{\epsilon, B}(s_1, \dots, s_\kappa) \ll \frac{1}{\delta^\kappa} \frac{(1 + B + |\sigma_1| + \dots + |\sigma_\kappa|)^{A(1+B+|\sigma_1|+\dots+|\sigma_\kappa|)}}{(1 + |t_1|)^{\frac{1}{2} - \sigma_1} \dots (1 + |t_\kappa|)^{\frac{1}{2} - \sigma_\kappa} (1 + |t_1| + \dots + |t_\kappa|)^{\sigma_1 + \dots + \sigma_\kappa - 1}}, \quad (7.7)$$

for some $A > 0$, provided that the s_i are located at a distance greater than $\delta > 0$ from the poles of $\Psi_{\epsilon, B}$.

Remark 6. *If $\epsilon = (-1, \dots, -1)$, then $\Psi_{\epsilon, B}$ has to be interpreted as being identically zero.*

Remark 7. *The function $\Psi_{\epsilon, B}(s_1, \dots, s_\kappa)$ has poles at $s_i \in \mathbb{Z}_{\leq 0}$ and at $s_1 + \dots + s_\kappa = B + 1$.*

Remark 8. *As a function G in this case we take $G(s) := \xi(\frac{1}{2} + s) / \xi(\frac{1}{2})$.*

We apply Lemma 8 to (7.5) with $\epsilon := (\text{sign } a_1, \dots, \text{sign } a_k)$, $v_1 = 1 - \alpha_1 - u_1 + \frac{w}{2}$ and $B = 4k$, so that by our choice for the lines of integration (7.6), we have $\Re(v_1) = 1 + 4k - \epsilon$. Notice that thanks to (7.7) we don't have problems of convergence of the integrals. We obtain,

$$K_{\alpha, \beta}(s) = \sum_{\substack{\nu=(\nu_2, \dots, \nu_k) \in \mathbb{Z}_{\geq 0}^{k-1}, \\ \nu_2 + \dots + \nu_k = 4k, \\ \nu_i = 0 \text{ if } \pm_i = -1}} \frac{(4k)!}{\nu_2! \cdots \nu_k!} (R_{\nu; \alpha, \beta}(s) - R'_{\nu; \alpha, \beta}(s))$$

where, after opening the Ramanujan sum and summing over n_2, \dots, n_k ,

$$\begin{aligned} R_{\nu; \alpha, \beta}(s) &:= \sum_{\ell \geq 1} \sum_{h \pmod{\ell}}^* \frac{1}{(2\pi i)^{k+1}} \int_{(c_w, c_{u_1}, c_{v_2}, \dots, c_{v_k})} \frac{|a_1|^{-1 + \alpha_1 + u_1 - \frac{w}{2}} N_1^{u_1 - s}}{(\ell/(a_1, \ell))^{1 - \alpha_1 + \beta_1 + w}} \tilde{P}(u_1 - s) \times \\ &\times \Psi_{\epsilon, 4k}(1 - \alpha_1 - u_1 + \frac{w}{2}, v_2, \dots, v_k) \zeta(1 - \alpha_1 + \beta_1 + w) \frac{G_{\alpha_1, \beta_1}(w)}{w} dw du_1 \\ &\times \prod_{i=2}^k \frac{|a_i|^{\nu_i - v_i}}{2\pi i} \int_{(c_{u_i})} D_{\alpha_i, \beta_i}(u_i + v_i - \nu_i, \frac{ha_i}{\ell}) \tilde{P}(u_i - s) N_i^{u_i - s} du_i dv_i \end{aligned}$$

with $c_{v_2}, \dots, c_{v_k} = \frac{\epsilon}{k}$, and where $R'_{\nu; \alpha, \beta}(s)$ is defined in the same way, but with lines of integrations $c'_{v_i} = 1/2 + \epsilon - \min(\Re(\alpha_i), \Re(\beta_i))$ for $i = 2, \dots, k$. We notice that by our choices for the lines of integration we have that the sum of the arguments of the function $\Psi_{\epsilon, 4k}$ in $R_{\nu; \alpha, \beta}(s)$ has real part $4k + 1 - \frac{\epsilon}{k}$ and so is less than $4k + 1$ as needed for the application of Lemma 8 (whereas for $R'_{\nu; \alpha, \beta}(s)$ one has that such real part is (much) larger than $4k + 1$).

Now, $R'_{\nu; \alpha, \beta}$ can be bounded trivially by moving the line of integrations c_{u_i} to $c_{u_i} = \frac{1}{2} + \nu_i$ for $i = 2, \dots, k$, c_w to $c_w = 1 + |\Re(\alpha_1 - \beta_1)| + 2\epsilon$, and c_{u_1} to $c_{u_1} = 1 - 6k - \Re(\alpha_1) + \frac{c_w}{2}$ without passing through any pole (cf. Remark 7). We obtain that $R'_{\nu; \alpha, \beta}(s)$ is bounded by

$$\begin{aligned} R'_{\nu; \alpha, \beta}(s) &\ll_{\epsilon} \left(\frac{k}{\epsilon}(1 + |s|)\right)^{Ak} \frac{N_2^{\frac{1}{2} + \nu_2 - \sigma} \cdots N_k^{\frac{1}{2} + \nu_k - \sigma} |a_2|^{\nu_2 - \frac{1}{2}} \cdots |a_k|^{\nu_k - \frac{1}{2}}}{N_1^{6k - 1 + \frac{1}{2} |\Re(\alpha_1 + \beta_1)| - \frac{\epsilon}{2} + \sigma}} \int_{(c_w, c_{u_1}, c_{v_2}, \dots, c_{v_k})} |G_{\alpha_1, \beta_1}(w)| \\ &\times |a_1|^{-1 + \Re(\alpha_1 + u_1 - \frac{w}{2})} |\tilde{P}(u_1 - s) \Psi_{\epsilon, 4k}(1 - \alpha_1 - u_1 + \frac{w}{2}, v_2, \dots, v_k)| dw du_1 dv_2 \cdots dv_k, \\ &\ll (1 + |\Im(\alpha_1)|)^{6k} X_{s, k, \mathbf{a}, \epsilon}^{Ak} \frac{N_2^{\frac{1}{2} + \nu_2 - \sigma} \cdots N_k^{\frac{1}{2} + \nu_k - \sigma}}{N_1^{6k - 1 + \frac{1}{2} |\Re(\alpha_1 + \beta_1)| - \frac{\epsilon}{2} + \sigma}} \ll \frac{(1 + |\Im(\alpha_1)|)^{6k} X_{s, k, \mathbf{a}, \epsilon}^{Ak} N_1^{-k}}{(N_1 \cdots N_k)^{\sigma - \frac{1}{2}}} \end{aligned}$$

by (6.4) and (7.7) with $X_{s, k, \mathbf{a}, \epsilon}$ as in (6.1), and where for the last bound we used that N_1 is the largest of the N_i and $\nu_2 + \dots + \nu_k = 4k$. Thus, summarizing this section we proved

$$A_{\alpha, \beta}(s) = \sum_{\substack{\{\alpha_1^*, \beta_1^*\} = \{\alpha_1, \beta_1\}, \\ (\alpha_i^*, \beta_i^*) = (\alpha_i, \beta_i) \quad \forall i \neq 1}} \sum_{\substack{\nu=(\nu_2, \dots, \nu_k) \in \mathbb{Z}_{\geq 0}^{k-1}, \\ \nu_2 + \dots + \nu_k = 4k, \\ \nu_i = 0 \text{ if } \pm_i = -1}} \frac{(4k)!}{\nu_2! \cdots \nu_k!} R_{\nu; \alpha^*, \beta^*}(s) + E_3(s), \quad (7.8)$$

where $E_3(s)$ satisfies

$$E_3(s) \ll Y_{\alpha,\beta}^{6k} X_{s,k,\mathbf{a},\varepsilon}^{Ak} N_1^{-k} (N_1 \cdots N_k)^{\frac{1}{2}-\sigma} \quad (7.9)$$

with $Y_{\alpha,\beta}$ as in (6.1).

7.2 Picking up the poles of the Estermann functions

Next, after moving c_w and c_{u_1} to ensure the convergence of the sum over ℓ , we move the line of integration c_{u_i} to $c_{u_i} = -\max(\Re(\alpha_i), \Re(\beta_i)) - 2\frac{\varepsilon}{k} + \nu_i$ for each $i = 2, \dots, k$, passing through the poles (cf. Lemma 2) of the Estermann functions. We obtain:

$$R_{\nu;\alpha,\beta}(s) = \sum_{\substack{I \cup J = \{2, \dots, k\}, \\ I \cap J = \emptyset}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i^*, \beta_i^*\} \ \forall i \in I, \\ (\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j) \ \forall j \in J \cup \{1\}}} \prod_{i \in I} \zeta(1 - \alpha_i + \beta_i) S_{I;\nu;\alpha^*,\beta^*}(s), \quad (7.10)$$

where after changing the order of sums and integrals (as can be done by the absolute convergence of the sum and integrals)

$$\begin{aligned} S_{I;\nu;\alpha,\beta}(s) &:= \frac{1}{(2\pi i)^{k+1}} \int_{(c_w, c_{u_1}, c_{v_2}, \dots, c_{v_k})} \Psi_{\varepsilon, 4k}(1 - \alpha_1 - u_1 + \frac{w}{2}, v_2, \dots, v_k) \zeta(1 - \alpha_1 + \beta_1 + w) \\ &\times \frac{G_{\alpha_1, \beta_1}(w)}{w} \tilde{P}(u_1 - s) N_1^{u_1 - s} du_1 \left(\prod_{i \in I} \tilde{P}(1 - \alpha_i - v_i + \nu_i - s) N_i^{1 - \alpha_i - v_i + \nu_i - s} dv_i \right) \\ &\times |a_1|^{-1 + \alpha_1 + u_1 - \frac{w}{2}} |a_2|^{\nu_2 - v_2} \cdots |a_k|^{\nu_k - v_k} \left(\prod_{j \in J} \frac{1}{2\pi i} \int_{(c_{u_j})} \tilde{P}(u_j - s) N_i^{u_j - s} \right) \\ &\times \sum_{\ell} \frac{(\ell, a_1)^w \prod_{i \in I \cup \{1\}} (a_i, \ell)^{1 - \alpha_i + \beta_i}}{\ell^{w + \sum_{i \in I \cup \{1\}} (1 - \alpha_i + \beta_i)}} dw \sum_{h \pmod{\ell}}^* \prod_{j \in J} D_{\alpha_j, \beta_j}(u_j + v_j - \nu_j, \frac{ha_j}{\ell}) du_j dv_j \end{aligned} \quad (7.11)$$

and the lines of integrations can be taken to be

$$\begin{aligned} c_{v_2} = \cdots = c_{v_k} &= \frac{\varepsilon}{k}, \quad c_{u_j} = -\max(\Re(\alpha_j), \Re(\beta_j)) - 2\frac{\varepsilon}{k} + \nu_j \ \forall j \in J, \\ c_w &= 2\varepsilon + \max\left(0, 1 + |J| - |I| + \sum_{i=1}^k |\Re(\alpha_i - \beta_i)|\right), \\ c_{u_1} &= -4k - \Re(\alpha_1) + \frac{c_w}{2} + \varepsilon. \end{aligned} \quad (7.12)$$

Notice that with this choice the sum over ℓ converges absolutely by the convexity bound (2.4) for the Estermann function.

We will treat $S_{I;\nu;\alpha,\beta}$ differently depending on whether $|I| \leq |J|$ or $|I| > |J|$.

7.3 The case $|I| \leq |J|$

If $|I| \leq |J|$ (or, equivalently, $|J| \geq \frac{k-1}{2}$), then we use the following lemma, whose proof we postpone until the end of this subsection.

Lemma 9. *Let $\mathbf{a} \in \mathbb{Z}_{\neq 0}^\kappa$ and $\boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{C}^\kappa$. Let \mathfrak{S} be the meromorphic function defined by*

$$\mathfrak{S}(z) := \sum_{h \pmod{\ell}}^* \prod_{j=1}^{\kappa} D_{\gamma_j, \delta_j} \left(z, \frac{ha_j}{\ell} \right).$$

Then

$$\mathfrak{S}(z) = \mathfrak{S}^*(z) + \mathfrak{S}^{**}(z)$$

where $\mathfrak{S}^*(z)$ is holomorphic on $\Re(z) \leq -\max_{i=1}^{\kappa} \max(\Re(\gamma_i), \Re(\delta_i))$ and $\mathfrak{S}^{**}(z) = 0$ if $\kappa = 1$ and otherwise $\mathfrak{S}^{**}(z)$ is holomorphic on $\Re(z) \leq \frac{1}{\kappa} - \frac{\varepsilon}{\kappa} - \frac{1}{\kappa} \sum_{i=1}^{\kappa} \max(\Re(\gamma_i), \Re(\delta_i))$. Furthermore, if $|\Re(\gamma_i)|, |\Re(\delta_i)| \leq \frac{1}{2}$ and

$$\Re(z) \leq -\frac{\varepsilon}{3\kappa} - \max(\Re(\gamma_i), \Re(\delta_i)) \quad \forall i = 1, \dots, \kappa, \quad u \geq 1 + \kappa - 2\kappa\Re(z) - \sum_{j=1}^{\kappa} \Re(\gamma_j + \delta_j) + \varepsilon,$$

then

$$\sum_{\ell \geq 1} \ell^{-u} |\mathfrak{S}^*(z)| \ll_{\varepsilon} |a_1 \cdots a_{\kappa}| \left(\frac{A\kappa}{\varepsilon} \right)^{4\kappa} (1 + A|\Re(z)|)^{2\kappa|\Re(z)|} \prod_{i=1}^{\kappa} (1 + |\gamma_i| + |\delta_i| + |\mathfrak{S}(z)|)^{2-2\Re(z)}. \quad (7.13)$$

Moreover, if $|\Re(\gamma_i)|, |\Re(\delta_i)| \leq \frac{1}{2(\kappa-1)}$ and

$$\Re(z) \leq \frac{1}{\kappa} - \frac{\varepsilon}{\kappa} - \frac{1}{\kappa} \sum_{i=1}^{\kappa} \max(\Re(\gamma_i), \Re(\delta_i)), \quad \Re(u) \geq 2 + \kappa - 2\kappa\Re(z) - \sum_{j=1}^{\kappa} \Re(\gamma_j + \delta_j) + \varepsilon,$$

then

$$\sum_{\ell \geq 1} \ell^{-u} |\mathfrak{S}^{**}(z)| \ll_{\varepsilon} \left(\frac{A\kappa}{\varepsilon} \right)^{4\kappa} (1 + A|\Re(z)|)^{2\kappa|\Re(z)|} \prod_{i=1}^{\kappa} (1 + |\gamma_i| + |\delta_i| + |\mathfrak{S}(z)|)^{2-2\Re(z)}. \quad (7.14)$$

We apply this Lemma with $\kappa = |J|$, splitting $S_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}(s)$ into

$$S_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}(s) = S_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}^*(s) + S_{I;\nu;\boldsymbol{\alpha},\boldsymbol{\beta}}^{**}(s)$$

in the way suggested by the notation, with $S^{**} = 0$ if $|J| = 1$. For S^* we use (7.13) with

$$z = 0, \quad \gamma_j = \alpha_j + u_j + v_j - \nu_j, \quad \delta_j = \beta_j + u_j + v_j - \nu_j$$

(since $D_{\alpha,\beta}(w, \frac{h}{k}) = D_{\alpha+w,\beta+w}(0, \frac{h}{k})$). We move the line of integration c_w to

$$c_w = 2\varepsilon + |J| - |I| + \sum_{i=1}^k |\Re(\alpha_i - \beta_i)|$$

keeping the other ones as in (7.12). Notice that we stay in the region of holomorphicity of \mathfrak{S}^* and that we can apply (7.13) since $-\max(\Re(\gamma_i), \Re(\delta_i)) = \frac{\varepsilon}{k} \geq \frac{\varepsilon}{3|J|}$ whereas, using the trivial bound $(a_j, \ell) \leq |a_j|$, the condition on u in the Lemma becomes

$$\Re(w) + \sum_{i \in I \cup \{1\}} \Re(1 - \alpha_i + \beta_i) \geq 1 + |J| - \sum_{j \in J} \Re(2u_j + 2v_j - 2\nu_j + \alpha_j + \beta_j) + \varepsilon$$

which is verified with our choice of lines of integration. Thus, we have that (7.13) gives

$$\begin{aligned} S^* &\ll_{\varepsilon} \left(\frac{k}{\varepsilon}\right)^{Ak} |a_2|^{2+\nu_2} \cdots |a_k|^{2+\nu_k} \int_{(c_w, c_{u_1}, c_{v_2}, \dots, c_{v_k})} |\Psi_{\varepsilon, 4k}(\frac{1}{2} - \alpha_1 - u_1 + \frac{w}{2}, v_2, \dots, v_k)| \\ &\quad \times |G_{\alpha_1, \beta_1}(w) \tilde{P}(u_1 - s) N_1^{u_1 - s}| |dw du_1| \left(\prod_{i \in I} |\tilde{P}(1 - \alpha_i - v_i + \nu_i - s) N_i^{1 - \alpha_i - v_i + \nu_i - s}| |dv_i| \right) \\ &\quad \times |a_1|^{-1 + \Re(\alpha_1 + u_1 + \frac{w}{2})} \prod_{j \in J} \int_{(c_{u_j})} |\tilde{P}(u_j - s) N_j^{u_j - s}| (1 + |\alpha_j| + |\beta_j| + |u_j - \nu_j + v_j|)^{2 + \frac{2+\varepsilon}{k}} |du_j dv_j|. \end{aligned}$$

Using (6.4) (with $r = 5$ and $r = 5k$), (7.4) (with $r = 5k$) and (7.7), we obtain

$$\begin{aligned} S^* &\ll_{\varepsilon} \left(\frac{k}{\varepsilon}\right)^{Ak} \max_{i=1}^k |a_i|^{Ak} (N_1 \cdots N_k)^{-\sigma} N_1^{-4k + \frac{1}{2}(|J| - |I|) + \eta_{\alpha, \beta} + 2\varepsilon} \left(\prod_{j \in J} N_j^{\nu_j} \right) \prod_{i \in I} N_i^{1 + \nu_i} \\ &\quad \times \int_{(\dots)} \frac{(1 + |\alpha_1 + u_1 - \frac{w}{2}|)^{4k - \frac{1}{2}} (1 + |s + u_1|)^{-5k} \prod_{i \in I} (1 + |s + v_i + \alpha_i|)^{-5}}{(1 + |w|)^{5k} (1 + |v_2|)^{\frac{1}{2} - \frac{\varepsilon}{k}} \cdots (1 + |v_k|)^{\frac{1}{2} - \frac{\varepsilon}{k}} (1 + |\alpha_1 + u_1 - \frac{w}{2}| + |v_2| + \cdots + |v_k|)^{4k - 1}} \\ &\quad \times \prod_{j \in J} \frac{(1 + |\alpha_j| + |\beta_j| + |u_j - \nu_j + v_j|)^3}{(1 + |s + u_j|)^5} |dv_2 \cdots dv_k dw| \prod_{j \in J \cup \{1\}} |du_j| \\ &\ll_{\varepsilon} X_{s, k, \mathbf{a}, \varepsilon}^{Ak} Y_{\alpha, \beta}^{7k} (N_1 \cdots N_k)^{-\sigma} N_1^{-4k + \frac{1}{2}(|J| - |I|) + \eta_{\alpha, \beta} + 2\varepsilon} \left(\prod_{j \in J} N_j^{\nu_j} \right) \prod_{i \in I} N_i^{1 + \nu_i}, \end{aligned}$$

where $\eta_{\alpha, \beta}$ is as in (2.1) and we used

$$\frac{1}{2} \sum_{i=1}^k |\Re(\alpha_i - \beta_i)| - \sum_{i \in I} \min(\Re(\alpha_i), \Re(\beta_i)) - \sum_{j \in J} \max(\Re(\alpha_j), \Re(\beta_j)) \leq 2\eta_{\alpha, \beta}.$$

Now, since $|J| = k - 1 - |I|$, the above bound implies

$$S^* \ll_{\varepsilon} X_{s,k,a,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} \frac{N_1^{\frac{k-1}{2} + \eta_{\alpha,\beta} + 2\varepsilon}}{(N_1 \cdots N_k)^{\sigma}} \left(\prod_j \left(\frac{N_j}{N_1} \right)^{\nu_j} \right) \prod_i \left(\frac{N_i}{N_1} \right)^{1 + \nu_i} \ll_{\varepsilon} X_{s,k,a,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} \frac{N_1^{\frac{k-1}{2} + \eta_{\alpha,\beta} + 2\varepsilon}}{(N_1 \cdots N_k)^{\sigma}},$$

since $\nu_2 + \cdots + \nu_k = 4k$ and $N_i \leq N_1$ for all $i = 2, \dots, k$.

If $|I| < |J|$ we can bound S^{**} in the same way, using (7.14) instead of (7.13), with the difference that now we move the line of integrations to

$$c_{u_j} = \nu_j + \frac{1}{|J|} - 2\frac{\varepsilon}{k} - \frac{1}{|J|} \sum_{r \in J} \max(\Re(\alpha_r), \Re(\beta_r)), \quad \forall j \in J, \quad c_{v_2} = \cdots = c_{v_k} = \frac{\varepsilon}{k},$$

$$c_w = -1 + |J| - |I| + \sum_{i=1}^k |\Re(\alpha_j + \beta_j)| + 3\varepsilon, \quad c_{u_1} = -4k - \Re(\alpha_1) + \frac{c_w}{2} + \varepsilon. \quad (7.15)$$

(Notice that, with respect to the case of S^* , we have essentially moved c_{u_j} to the right by $\frac{1}{|J|}$, c_w to the left by 1 and thus c_{u_1} to the right by $\frac{1}{2}$). Thus, we obtain

$$S^{**} \ll_{\varepsilon} X_{s,k,a,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} \frac{N_1^{\frac{k}{2} - 1 + \eta_{\alpha,\beta} + 2\varepsilon}}{(N_1 \cdots N_k)^{\sigma}} \left(\prod_{j \in J} \left(\frac{N_j}{N_1} \right)^{\nu_j} \right) \left(\prod_{i \in I} \left(\frac{N_i}{N_1} \right)^{1 + \nu_i} \right) \prod_{j \in J} N_j^{\frac{1}{|J|}}.$$

Also, we have

$$\left(\prod_{i \in I} \frac{N_i}{N_1} \right) \prod_{j \in J} N_j^{\frac{1}{|J|}} = N_1^{-\frac{|I|+1}{|J|}} \prod_{i \in I} \left(\frac{N_i}{N_1} \right)^{1 - \frac{1}{|J|}} \prod_{i=1}^k N_i^{\frac{1}{|J|}} \leq N_1^{-\frac{|I|+1}{|J|}} \prod_{i=1}^k N_i^{\frac{1}{|J|}} = N_1^{1 - \frac{k}{|J|}} \prod_{i=1}^k N_i^{\frac{1}{|J|}},$$

since $|I| = k - 1 - |J|$, and the maximum value of the expression on the right is obtained for $|J| = k - 1$ since $N_1^k \geq N_1 \cdots N_k$. Thus,

$$S^{**} \ll_{\varepsilon} X_{s,k,a,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{\frac{k}{2} - 1 - \frac{1}{k-1} + \eta_{\alpha,\beta} + 2\varepsilon} (N_1 \cdots N_k)^{\frac{1}{k-1} - \sigma}.$$

If $|I| = |J|$, then we cannot move the line of integration c_w as in (7.15) without passing through the pole at $w = 0$. Thus, we move c_w to $c_w = \sum_{i=1}^k |\Re(\alpha_j + \beta_j)| + 2\varepsilon$ and leave the other lines of integrations as in (7.15). Bounding trivially we obtain

$$S^{**} \ll_{\varepsilon} X_{s,k,a,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{\eta_{\alpha,\beta} - \sigma + 2\varepsilon} \prod_{i=2}^k \left(\frac{N_i}{N_1} \right)^{\nu_i} \prod_{i \in I} N_i^{1 - \sigma} \prod_{j \in J} N_j^{\frac{1}{|J|} - \sigma}.$$

Now we have $\frac{1}{|J|} - \sigma \leq 0$ for $\sigma \geq \frac{1}{2}$ (we can take $|J| > 1$ since otherwise $S^{**} = 0$) and $N_1 \geq N_i$ for all $i = 2, \dots, k$ and so for $\frac{1}{2} \leq \sigma \leq 1$ we have

$$S^{**} \ll_{\varepsilon} X_{s,k,a,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{\eta_{\alpha,\beta} - \sigma + 2\varepsilon + |I|(1 - \sigma)} = X_{s,k,a,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{\frac{k-1}{2} - \frac{k+1}{2}\sigma + \eta_{\alpha,\beta} + 2\varepsilon}.$$

since in this case $|I| = \frac{k-1}{2}$. Thus, summarizing, in this subsection we proved that if $|J| \geq |I|$ and $\frac{1}{2} \leq \sigma \leq 1$, then

$$S_{I;\nu;\alpha,\beta}(s) \ll_{\varepsilon} X_{s,k,\alpha,\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{-1+\eta_{\alpha,\beta}+2\varepsilon} \left(\frac{N_1^{\frac{k+1}{2}} + N_1^{\frac{k}{2}} (N_2 \cdots N_k)^{\frac{1}{k-1}}}{(N_1 \cdots N_k)^{\sigma}} + N_1^{\frac{k+1}{2}(1-\sigma)} \right). \quad (7.16)$$

We conclude the subsection with the proof of Lemma 9.

Proof of Lemma 9. First, we write $\ell_j = \ell/(a_j, \ell)$ and $a'_j = a_j/(a_j, \ell)$, so that

$$\mathfrak{S}(z) := \sum_{h \pmod{\ell}}^* \prod_{j=1}^{\kappa} D_{\gamma_j, \delta_j} \left(z, \frac{ha'_j}{\ell_j} \right).$$

We apply the functional equation (2.2) to each of the Estermann functions, obtaining

$$\mathfrak{S}(z) = \sum_{\eta=(\eta_1, \dots, \eta_{\kappa}) \in \{\pm 1\}^{\kappa}} \ell_j^{1-2z-\gamma_j-\delta_j} \chi_{\eta_j}(z; \gamma_j, \delta_j) \sum_{h \pmod{\ell}}^* D_{-\gamma_j, -\delta_j} \left(1-z, \eta_j \frac{\overline{ha'_j}}{\ell_j} \right),$$

where $\overline{ha'_j}$ is the inverse of $ha'_j \pmod{\ell_j}$. Now we assume $\Re(z) < -\max_{i=1}^{\kappa} \max(\Re(\gamma_i), \Re(\delta_i))$ and we expand the Estermann functions into their Dirichlet series and execute the sum over h . We obtain

$$\begin{aligned} \mathfrak{S}(z) &= \sum_{\eta \in \{\pm 1\}^{\kappa}} \ell_j^{1-2z-\gamma_j-\delta_j} \chi_{\eta_j}(z; \gamma_j, \delta_j) \\ &\quad \times \sum_{m_1, \dots, m_{\kappa}} \frac{\tau_{-\gamma_1, -\delta_1}(m_1) \cdots \tau_{-\gamma_{\kappa}, -\delta_{\kappa}}(m_{\kappa})}{m_1^{1-z} \cdots m_{\kappa}^{1-z}} c_{\ell}(\eta_1 \overline{a'_1}(a_1, \ell) m_1 + \cdots + \eta_{\kappa} \overline{a'_{\kappa}}(a_{\kappa}, \ell) m_{\kappa}) \\ &= \sum_{\eta \in \{\pm 1\}^{\kappa}} \ell_j^{1-2z-\gamma_j-\delta_j} \chi_{\eta_j}(z; \gamma_j, \delta_j) \sum_{m_1, \dots, m_{\kappa}} \frac{\tau_{-\gamma_1, -\delta_1}(m_1) \cdots \tau_{-\gamma_{\kappa}, -\delta_{\kappa}}(m_{\kappa})}{m_1^{1-z} \cdots m_{\kappa}^{1-z}} c_{\ell}(\rho) \end{aligned}$$

where

$$\rho := \eta_1(a_1, \ell) m_1 \frac{a'_1 \cdots a'_{\kappa}}{a'_1} + \cdots + \eta_{\kappa}(a_{\kappa}, \ell) m_{\kappa} \frac{a'_1 \cdots a'_{\kappa}}{a'_{\kappa}}.$$

Then we divide $\mathfrak{S}(z)$ into $\mathfrak{S}(z) = \mathfrak{S}^*(z) + \mathfrak{S}^{**}(z)$ according to whether $\rho \neq 0$ or $\rho = 0$ (notice that $\mathfrak{S}^{**}(z) = 0$ if $\kappa = 1$). For the terms with $\rho = 0$, we observe that by Lemma 4 we have

$$\sum_{\substack{m_1, \dots, m_{\kappa} \geq 1 \\ \rho=0}} \frac{\tau_{-\gamma_1, -\delta_1}(m_1) \cdots \tau_{-\gamma_{\kappa}, -\delta_{\kappa}}(m_{\kappa})}{m_1^{1-z} \cdots m_{\kappa}^{1-z}} \ll_{\varepsilon} \left(\frac{A\kappa}{\varepsilon} \right)^{4\kappa}$$

for

$$\Re(z) \leq \frac{1}{\kappa} - \frac{\varepsilon}{\kappa} - \frac{1}{\kappa} \sum_{i=1}^{\kappa} \max(\Re(\gamma_i), \Re(\delta_i)).$$

Thus from Stirling's formula in the crude form

$$\Gamma(\Re(z) + it) \ll c^{-1} (1 + A|\Re(z)|)^{|\Re(z)|} (1 + |\Im(z)|)^{\Re(z) - \frac{1}{2}} e^{-\frac{\pi}{2}|\Im(z)|}, \quad \Re(z) \geq c > 0,$$

and since $\ell_j \leq \ell$, for $\Im(\gamma_j), \Im(\delta_j) \ll 1$ and $|\Re(\gamma_i)|, |\Re(\delta_i)| \leq \frac{1}{2}$ we have

$$\begin{aligned} \mathfrak{G}^{**}(z) &\ll_{\varepsilon} \ell^{\kappa+1-2\kappa\Re(z)-\sum_{j \in J} \Re(\gamma_j+\delta_j)} \left(\frac{A\kappa}{\varepsilon}\right)^{4\kappa} \sum_{\eta \in \{\pm 1\}^{\kappa}} |\chi_{\eta_j}(z; \gamma_j, \delta_j)| \\ &\ll_{\varepsilon} \ell^{\kappa+1-2\kappa\Re(z)-\sum_{j \in J} \Re(\gamma_j+\delta_j)} \left(\frac{A\kappa}{\varepsilon}\right)^{4\kappa} (1 + A|\Re(z)|)^{2\kappa|\Re(z)|} \prod_{j=1}^{\kappa} (1 + |\Im(z)|)^{1-2\Re(z)-\Re(\gamma_j+\delta_j)} \end{aligned}$$

and (7.14) follows.

In order to prove (7.13) it is sufficient to show that for all fixed $\varepsilon > 0$ we have

$$Z := \sum_{\ell} \ell^{-u'} \left| \sum_{\substack{m_1, \dots, m_{\kappa}, \\ \rho \neq 0}} \frac{\tau_{-\gamma_1, -\delta_1}(m_1) \cdots \tau_{-\gamma_{\kappa}, -\delta_{\kappa}}(m_{\kappa})}{m_1^{1-z} \cdots m_{\kappa}^{1-z}} c_{\ell}(\rho) \right| \ll |a_1 \cdots a_{\kappa}| (A\kappa/\varepsilon)^{4\kappa} \quad (7.17)$$

for $\Re(z) \leq -\frac{\varepsilon}{3\kappa} - \max_i(\max(\gamma_i, \delta_i))$ $u' \geq 1 + \varepsilon$. Moreover, we observe that we can assume z, γ_i, δ_i are real and we can also assume $z = -\frac{\varepsilon}{3\kappa} - \max_i(\max(\gamma_i, \delta_i))$.

Since $|c_{\ell}(\rho)| \leq \sum_{d|(\rho, \ell)} d \ll_{\varepsilon} (\rho, \ell)^{1+\varepsilon}$, we have

$$Z \ll_{\varepsilon} \sum_{m_1, \dots, m_{\kappa}} \frac{\tau_{-\gamma_1, -\delta_1}(m_1) \cdots \tau_{-\gamma_{\kappa}, -\delta_{\kappa}}(m_{\kappa})}{m_1^{1-z} \cdots m_{\kappa}^{1-z}} \sum_{\substack{\ell \geq 1, \\ \rho \neq 0}} \frac{(\ell, \rho)^{1+\varepsilon}}{\ell^u}.$$

(Notice that ρ depends on ℓ .) Now, we write

$$\rho' := \eta_1 d_1 m_1 \frac{a_1 \cdots a_{\kappa} d_1}{a_1 d_1 \cdots d_{\kappa}} + \cdots + \eta_{\kappa} d_{\kappa} m_{\kappa} \frac{a_1 \cdots a_{\kappa} d_{\kappa}}{a_{\kappa} d_1 \cdots d_{\kappa}}$$

for $d_i = (a_i, \ell)$, so that

$$\sum_{\substack{\ell \geq 1, \\ \rho \neq 0}} \frac{(\ell, \rho)^{1+\varepsilon}}{\ell^u} = \sum_{\substack{d_1 | a_1, \dots, d_{\kappa} | a_{\kappa}, \\ \rho' \neq 0}} \sum_{\substack{\ell, \\ d_i | \ell, (\ell, a_i/d_i)=1}} \frac{(\ell, \rho')^{1+\varepsilon}}{\ell^u} \ll_{\varepsilon} \frac{1}{u-1} \sum_{\substack{d_1 | a_1, \dots, d_{\kappa} | a_{\kappa}, \\ \rho' \neq 0}} \sigma_{1-u+\varepsilon}(\rho'),$$

where $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$, since for all $c \in \mathbb{N}$, $u > 1$ we have

$$\sum_{\ell \geq 1} \frac{(\ell, c)^{1+\varepsilon}}{\ell^u} = \sum_{d|c} d^{1+\varepsilon} \sum_{\substack{d|\ell, \\ (c/d, \ell)=1}} \frac{1}{\ell^u} \ll \zeta(1+u) \sum_{d|c} d^{1-u+\varepsilon} \ll_\varepsilon \frac{1}{u-1} \sigma_{1-u+\varepsilon}(c).$$

Thus, for $u \geq 1 + \varepsilon$

$$Z \ll_\varepsilon \frac{1}{u-1} \sum_{d_1|a_1, \dots, d_\kappa|a_\kappa} \sum_{\substack{m_1, \dots, m_\kappa \geq 1 \\ \rho' \neq 0}} \frac{\tau_{-\gamma_1, -\delta_1}(m_1) \cdots \tau_{-\gamma_\kappa, -\delta_\kappa}(m_\kappa)}{m_1^{1-z} \cdots m_\kappa^{1-z}} d(\rho'). \quad (7.18)$$

We divide the sum into a sum of k sums, according to which of the m_i is the largest. For the contribution where $m_1 \geq \max_{i=2}^\kappa(m_i)$ we observe that, for $z_1 := 1 - z - \max(\gamma_1, \delta_1) > 1$ we have

$$\begin{aligned} \sum_{\substack{m_1 \geq \max_{i=2}^\kappa(m_i), \\ \rho' \neq 0}} \frac{d(\rho') \tau_{-\gamma_1, -\delta_1}(m_1)}{m_1^{1-z}} &\ll \sum_{\substack{m_1 \geq \max_{i=2}^\kappa(m_i), \\ \rho' \neq 0}} \frac{d(\rho') d(m_1)}{m_1^{z_1}} \\ &\ll \left(\sum_{\substack{m_1 \geq \max_{i=2}^\kappa(m_i), \\ \rho' \neq 0}} \frac{d(\rho')^2}{m_1^{z_1}} \right)^{\frac{1}{2}} \left(\sum_{\substack{m_1 \geq \max_{i=2}^\kappa(m_i), \\ \rho' \neq 0}} \frac{d(m_1)^2}{m_1^{z_1}} \right)^{\frac{1}{2}}, \end{aligned}$$

by the Cauchy-Schwarz inequality. Then, since $|\rho'| \leq \kappa m_1 |a_1 \cdots a_\kappa|$ we have that the above sum is

$$\begin{aligned} &\leq \left(|\kappa a_1 \cdots a_\kappa|^{z_1} \sum_{\rho' \in \mathbb{Z} \setminus \{0\}} \frac{d(\rho')^2}{|\rho'|^{z_1}} \right)^{\frac{1}{2}} \left(\sum_{m_1 \geq 1} \frac{d(m_1)^2}{m_1^{z_1}} \right)^{\frac{1}{2}} \leq |\kappa m_1 a_1 \cdots a_\kappa|^{\frac{1}{2}} \left(2 \sum_{m_1 \geq 1} \frac{d(m_1)^2}{m_1^{z_1}} \right) \\ &\ll |\kappa m_1 a_1 \cdots a_\kappa|^{\frac{1}{2}} (z_1 - 1)^{-4} \end{aligned}$$

since $z_1 \leq 2$ and $\sum_{n \geq 1} \frac{d(n)^2}{n^{2s}} = \frac{\zeta(2s)^4}{\zeta(4s)}$. Thus, (7.18) can be bounded by

$$Z \ll_\varepsilon \frac{\kappa^2}{(u-1)} |a_1 \cdots a_\kappa| \prod_{i=1}^k (-z - \max(\gamma_i, \delta_i))^{-4}$$

and so (7.17) follows. \square

7.4 The case $|I| > |J|$

Here we treat $S_{I, \nu; \alpha, \beta}(s)$, which was defined in (7.11), in the case $|I| > |J|$. First, we move the lines of integration c_w and c_{u_1} in (7.11) to

$$\begin{aligned} c_w &= 2\varepsilon + |\Re(\alpha_1 - \beta_1)|, & c_{u_1} &= -4k - \Re(\alpha_1) + \frac{c_w}{2} + \varepsilon \\ c_{v_2} &= \cdots = c_{v_k} = \frac{\varepsilon}{k}, & c_{u_j} &= -\max(\Re(\alpha_j), \Re(\beta_j)) - 2\frac{\varepsilon}{k} + \nu_j \quad \forall j \in J. \end{aligned}$$

Bounding as before (in this case one could also simply use the convexity bound for the Estermann functions) one obtains

$$S_{I;\nu;\alpha,\beta}(s) \ll X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{-1+\eta_{\alpha,\beta}+2\varepsilon} (N_1 \cdots N_k)^{1-\sigma} \left(\prod_{i \in I} \left(\frac{N_i}{N_1} \right)^{\nu_i} \right) \prod_{j \in J} \frac{N_j^{\nu_j-1}}{N_1^{\nu_j}}.$$

In particular, if $\nu_j > 0$ for some $j \in J$, then the above bound implies

$$S_{I;\nu;\alpha,\beta}(s) \ll X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{-2+\eta_{\alpha,\beta}+2\varepsilon} (N_1 \cdots N_k)^{1-\sigma}.$$

For the terms with $\nu_j = 0$ for all $j \in J$, we take $j^* \in J$ and move the lines of integrations $c_{u_{j^*}}$ and $c_{v_{j^*}}$ to the left and to the right by 1 respectively, picking up the residue from the simple pole of $\Psi_{\varepsilon,4k}$ at $v_{j^*} = 0$. In the contribution coming from the integrals of the new lines of integration we move c_{u_1} to the right by 1 (as we can now do without passing through other poles of $\Psi_{\varepsilon,4k}$) so that bounding trivially we obtain a contribution which is

$$\begin{aligned} &\ll X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{-2+\eta_{\alpha,\beta}+2\varepsilon} (N_1 \cdots N_k)^{1-\sigma} \left(\prod_{i \in I} \left(\frac{N_i}{N_1} \right)^{\nu_i} \right) N_{j^*} \prod_{j \in J} N_j^{-1} \\ &\ll X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{-2+\eta_{\alpha,\beta}+2\varepsilon} (N_1 \cdots N_k)^{1-\sigma}. \end{aligned}$$

We repeat this for all $j \in J$ obtaining

$$S_{I;\nu;\alpha,\beta}(s) = T_{I;\nu;\alpha,\beta}(s) + O\left(X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} \frac{(N_1 \cdots N_k)^{1-\sigma}}{N_1^{2-\eta_{\alpha,\beta}-2\varepsilon}} \right) \quad (7.19)$$

where $T_{I;\nu;\alpha,\beta}(s)$ is obtained by $S_{I;\nu;\alpha,\beta}(s)$ by taking the residue in at $v_j = 0$ for all $j \in J$, that is

$$\begin{aligned} T_{I;\nu;\alpha,\beta}(s) &:= \frac{1}{(2\pi i)^{k+1}} \int_{(c_w, c_{u_1})} \left(\prod_{i \in I} \int_{(c_{v_i})} \right) \Psi'_{I_1, \varepsilon}(\mathbf{v}_{I_1}) \frac{\tilde{P}(u_1 - s) N_1^{u_1 - s}}{|a_1|^{1-\alpha_1 - u_1 + \frac{w}{2}}} \\ &\times \frac{G(B + \alpha_1 + u_1 - \frac{w}{2} - \sum_{i \in I} v_i)}{B + \alpha_1 + u_1 - \frac{w}{2} - \sum_{i \in I} v_i} \sum_{\ell \geq 1} \frac{\prod_{i \in I_1} (a_i, \ell)^{1-\alpha_i + \beta_i}}{\ell^{w + \sum_{i \in I_1} (1-\alpha_i + \beta_i)} (\ell, a_1)^{-w}} \frac{G_{\alpha_1, \beta_1}(w)}{w} du_1 \\ &\times \zeta(1 - \alpha_1 + \beta_1 + w) \left(\prod_{i \in I} \tilde{P}(1 - \alpha_i - v_i + \nu_i - s) N_i^{1-\alpha_i - v_i + \nu_i - s} |a_i|^{\nu_i - v_i} dv_i \right) dw \\ &\times \sum_{h \pmod{\ell}}^* \prod_{j \in J} \int_{(c_{u_j})} \tilde{P}(u_j - s) N_i^{u_j - s} D_{\alpha_j, \beta_j}(u_j, \frac{ha_j}{\ell}) du_j. \end{aligned}$$

with $I_1 := I \cup \{1\}$, $\mathbf{v}_{I_1} := (v_i)_{i \in I_1}$, where we put $v_1 := 1 - \alpha_1 - u_1 + \frac{w}{2}$ and

$$\Psi_{I_1, \varepsilon}(\mathbf{v}_{I_1}) := \frac{\prod_{i \in I_1} \Gamma(v_i)}{\Gamma(V_{I_1; \varepsilon}^+(\mathbf{v}_{I_1})) \Gamma(V_{I_1; \varepsilon}^-(\mathbf{v}_{I_1}))}, \quad V_{I_1; \varepsilon}^{\pm}(\mathbf{v}_{I_1}) := \sum_{\substack{i \in I_1, \\ \pm i = \pm 1}} v_i. \quad (7.20)$$

Notice that in the integral defining $T_{I;\nu;\alpha,\beta}(s)$ we have a fast decaying function for each of the variables of integration and so we don't have to worry anymore about the convergence of the integrals.

Next, we move the line of integrations to

$$c_{u_j} = 1 - \max(\alpha_j, \beta_j) - \frac{\varepsilon}{k} \quad \forall j \in J, \quad c_w = -|I| + \sum_{i=1}^k |\alpha_j - \beta_j| + 2\varepsilon$$

moving also c_{u_1} so that we still have $c_{u_1} = -4k - \Re(\alpha_1) + \frac{c_w}{2} + \varepsilon$. We pass through a simple pole at $w = 0$ only, since the pole of $\zeta(1 - \alpha_1 + \beta_1 + w)$ is canceled by the zero of $G_{\alpha_1, \beta_1}(w)$ (cf. Remark 5). Notice that on the new line of integrations the convexity bound (2.4) gives

$$D_{\alpha_j, \beta_j}(u_j, \frac{h a_j}{\ell}) \ll (k/\varepsilon)^2 (\ell(|u_j| + |\alpha_j|))^{\frac{1}{2}} (|u_j| + |\beta_j|)^{\frac{1}{2}} |\Re(\alpha_j - \beta_j)| + \frac{\varepsilon}{k}$$

which suffices for the convergence of the sum over ℓ . Thus the integral on the new lines of integrations gives a contribution bounded by

$$X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} N_1^{-|I|+\eta_{\alpha,\beta}+2\varepsilon} (N_1 \cdots N_k)^{1-\sigma}.$$

Thus, since $|I| > |J|$ with $|I| + |J| = k - 1 \geq 2$ implies $|I| \geq 2$, then by (7.19) we have

$$S_{I;\nu;\alpha,\beta}(s) = Y_{I;\nu;\alpha,\beta}(s) + O\left(X_{s,k,\mathbf{a},\varepsilon}^{Ak} Y_{\alpha,\beta}^{7k} (N_1 \cdots N_k)^{1-\sigma} N_1^{-2+\eta_{\alpha,\beta}+2\varepsilon}\right) \quad (7.21)$$

where $Y_{I;\nu;\alpha,\beta}(s)$ is the contribution from the residue at $w = 0$.

Now, we move the line of integration c_{u_1} to $c_{u_1} = 1 - \Re(\alpha_1) + \frac{\varepsilon}{k}$ and we make the change of variables $v_i \rightarrow v_i + \nu_i$ for all $i \in I$ moving the lines of integration c_{v_i} so that we still have $c_{v_i} = \frac{\varepsilon}{k}$ for all $i \in I$. Since $B = \sum_{i \in I} \nu_i$ (as we only have to consider the terms with $\nu_j = 0$ for all $j \in J \cup \{1\}$) we have that the only factor depending on ν is the function $\Psi'_{I_1, \varepsilon}(\mathbf{v}_i + \boldsymbol{\nu})$, where $\boldsymbol{\nu} = (0, \nu_2, \dots, \nu_k)$. Thus, summing over ν we are left with

$$\begin{aligned} & \sum_{\substack{\nu=(\nu_2, \dots, \nu_k) \in \mathbb{Z}_{\geq 0}^{k-1}, \\ \nu_2 + \dots + \nu_k = 4k, \\ \nu_i = 0 \text{ if } \pm i = -1 \text{ or } i \in J}} \frac{(4k)!}{\nu_2! \cdots \nu_k!} \Psi'_{I_1, \varepsilon}(\mathbf{v}_i + \boldsymbol{\nu}) \\ &= \frac{\prod_{i \in I_1^-} \Gamma(v_i)}{\Gamma(V_{I_1; \varepsilon}(\mathbf{v}_{I_1}))} \sum_{\substack{\nu_I = (\nu_i)_{i \in I_1^+}, \nu_i \in \mathbb{Z}_{\geq 0}, \\ \sum_{i \in I_1^+} \nu_i = 4k}} \frac{(4k)! \prod_{i \in I_1^+} \Gamma(v_i + \nu_i)}{\prod_{i \in I_1^+} \nu_i! \Gamma(\sum_{i \in I_1^+} (v_i + \nu_i))}, \end{aligned}$$

where $I_1^\pm := \{i \in I \mid \pm i = \pm 1\}$. The identity $B(s_1 + 1, s_2) + B(s_1, s_2 + 1) = B(s_1, s_2)$ for the Beta function $B(s_1, s_2) := \Gamma(s_1)\Gamma(s_2)\Gamma(s_1 + s_2)^{-1}$ generalizes to

$$\sum_{\substack{(r_1, \dots, r_m) \in \mathbb{Z}_{\geq 0}^m, \\ r_1 + \dots + r_m = r}} \frac{r!}{r_1! \cdots r_m!} \frac{\Gamma(s_1 + r_1) \cdots \Gamma(s_m + r_m)}{\Gamma(r + s_1 + \cdots + s_m)} = \frac{\Gamma(s_1) \cdots \Gamma(s_m)}{\Gamma(s_1 + \cdots + s_m)}, \quad (7.22)$$

for $m, r \geq 1$, $s_1, \dots, s_m \in \mathbb{C}$, and so we have

$$\sum_{\substack{\nu=(\nu_2, \dots, \nu_k) \in \mathbb{Z}_{\geq 0}^{k-1}, \\ \nu_2 + \dots + \nu_k = 4k, \\ \nu_i = 0 \text{ if } \pm_i = -1 \text{ or } i \in J}} \frac{(4k)!}{\nu_2! \cdots \nu_k!} \Psi'_{I_1, \epsilon}(\mathbf{v}_i + \boldsymbol{\nu}) = \frac{\prod_{i \in I_1} \Gamma(v_i)}{\Gamma(V_{I_1; \epsilon}^-(\mathbf{v}_{I_1})) \Gamma(V_{I_1; \epsilon}^+(\mathbf{v}_{I_1}))} = \Psi'_{I_1, \epsilon}(\mathbf{v}_i).$$

Thus, after the change of variables $u_j \rightarrow u_j + s$ for all $j \in J \cup \{1\}$ and $1 - \alpha_i - v_i - s \rightarrow u_i$ for $i \in I$, we obtain

$$\sum_{\substack{\nu=(\nu_2, \dots, \nu_k) \in \mathbb{Z}_{\geq 0}^{k-1}, \\ \nu_2 + \dots + \nu_k = 4k, \\ \nu_i = 0 \text{ if } \pm_i = -1 \text{ or } i \in J}} \frac{(4k)!}{\nu_2! \cdots \nu_k!} Y_{I; \nu; \alpha, \beta}(s) = \zeta(1 - \alpha_1 + \beta_1) Z_{I_1; \alpha, \beta}(s) \quad (7.23)$$

where for any set \mathcal{I} with $|\mathcal{I}| > \frac{k+1}{2}$ we define

$$\begin{aligned} Z_{\mathcal{I}; \alpha, \beta}(s) &= \frac{1}{(2\pi i)^k} \int_{(c_{u_1}, \dots, c_{u_k})} \left(\prod_{i=1}^k \tilde{P}(u_i) N_i^{u_i} \right) \Psi_{\mathcal{I}, \epsilon}(\mathbf{v}'_{\mathcal{I}}) \left(\prod_{i \in \mathcal{I}} |a_i|^{-1 + \alpha_i + u_i + s} \right) \frac{G(1 - \sum_{i \in I_1} v'_i)}{1 - \sum_{i \in I} v'_i} \\ &\times \sum_{\ell \geq 1} \frac{\prod_{i \in \mathcal{I}} (a_i, \ell)^{1 - \alpha_i + \beta_i}}{\ell^{\sum_{i \in \mathcal{I}} (1 - \alpha_i + \beta_i)}} \sum_{h \pmod{\ell}}^* \prod_{j \in \mathcal{J}} \left(D_{\alpha_j, \beta_j}(u_j + s, \frac{ha_j}{\ell}) \right) du_1 \cdots du_k \end{aligned} \quad (7.24)$$

where $\mathbf{v}'_{\mathcal{I}} = (v'_i)_{i \in \mathcal{I}} = (1 - \alpha_i - u_i - s)_{i \in \mathcal{I}}$, $\mathcal{J} := \{1, \dots, k\} \setminus \mathcal{I}$ and where we can take the lines of integrations to be

$$c_{u_j} = 1 - \max(\alpha_j, \beta_j) - \sigma - \frac{\varepsilon}{k} \quad \forall j \in \mathcal{J}, \quad c_{u_i} = 1 - \alpha_i - \sigma - \frac{\varepsilon}{k} \quad \forall i \in \mathcal{I}.$$

Notice that for any $i \in \mathcal{I}$ if we move $c_{u_i}^*$ to $-\alpha_i^* - \sigma + \varepsilon$ we obtain the bound

$$Z_{\mathcal{I}; \alpha, \beta}(s) \ll X_{s, k, \mathbf{a}, \varepsilon}^{Ak} Y_{\alpha, \beta}^k (N_1 \cdots N_k)^{1 - \sigma} N_{i^*}^{-1 + \eta_{\alpha, \beta} + \varepsilon}. \quad (7.25)$$

Also, for all $j^* \in \mathcal{J}$ (notice we can assume $k \geq 4$ since for $k = 3$ we have $|\mathcal{I}| > 2$ and so $\mathcal{J} = \emptyset$) we have

$$Z_{\mathcal{I}; \alpha, \beta}(s) \ll X_{s, k, \mathbf{a}, \varepsilon}^{Ak} Y_{\alpha, \beta}^k (N_1 \cdots N_k)^{1 - \sigma - \frac{1}{k}} (N_{j^*})^{-1 + \eta_{\alpha, \beta} + \varepsilon} \quad (7.26)$$

as can be seen by moving the lines of integrations to

$$\begin{aligned} c_{u_j} &= 1 - \max(\alpha_j, \beta_j) - \sigma - \frac{1}{k} \quad \forall j \in J \setminus \{j^*\}, & c_{u_i} &= 1 - \alpha_i - \sigma - \frac{1}{k} + \frac{\varepsilon}{k} \quad \forall i \in \mathcal{I}, \\ c_{u_{j^*}} &= -\sigma - \frac{1}{k} - \max(\alpha_{j^*}, \beta_{j^*}) + \eta_{\alpha, \beta}. \end{aligned}$$

(Notice that since $|\mathcal{I}| \geq \lceil \frac{k+3}{2} \rceil$, with this choice the sum over ℓ is absolutely convergent by the convexity bound (2.4).)

To summarize, by (7.21) and (7.23) in this section we proved that for $|I| > |J|$

$$\sum_{\substack{\nu=(\nu_2, \dots, \nu_k) \in \mathbb{Z}_{>0}^{k-1}, \\ \nu_2 + \dots + \nu_k = 4k, \\ \nu_i = 0 \text{ if } \pm_i = -1}} \frac{(4k)!}{\nu_2! \cdots \nu_k!} S_{I; \nu; \alpha, \beta}(s) = \zeta(1 - \alpha_1 + \beta_1) Z_{I_1; \alpha, \beta}(s) + E_4(s) \quad (7.27)$$

where $E_4(s)$ satisfies

$$E_4(s) \ll X_{s, k, \mathbf{a}, \varepsilon}^{Ak} Y_{\alpha, \beta}^{7k} (N_1 \cdots N_k)^{1-\sigma} N_1^{-2+\eta_{\alpha, \beta}+2\varepsilon}$$

and $Z_{I_1; \alpha, \beta}(s)$ satisfies (7.25) and (7.26).

7.5 Conclusion of the proof of Lemma 5

By (7.8), (7.9), (7.10), (7.16) and (7.27)

$$A_{\alpha, \beta}(s) = \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, \\ 1 \in \mathcal{I}, |\mathcal{I}| > \frac{k+1}{2}}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i^*, \beta_i^*\} \ \forall i \in \mathcal{I}, \\ (\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j) \ \forall j \in \mathcal{J}}} \prod \zeta(1 - \alpha_i + \beta_i) Z_{\mathcal{I}; \alpha^*, \beta^*}(s) + E_5(s), \quad (7.28)$$

where $E_5(s)$ is an entire function of s satisfying

$$E_5(s) \ll X_{s, k, \mathbf{a}, \varepsilon}^{Ak} Y_{\alpha, \beta}^{7k} N_1^{\eta_{\alpha, \beta}+2\varepsilon} \left(\frac{N_1^{\frac{k-1}{2}}}{(N_1 \cdots N_k)^\sigma} + \frac{N_1^{\frac{k}{2}-1-\frac{1}{k-1}}}{(N_1 \cdots N_k)^{\sigma-\frac{1}{k-1}}} + \frac{(N_1 \cdots N_k)^{1-\frac{1}{k}-\sigma}}{N_1} + N_1^{\frac{k-1}{2}-\frac{k+1}{2}\sigma} \right)$$

for $\frac{1}{2} \leq \sigma \leq 1$. Thus, to obtain (6.6) and (6.7) we just need to extend the sum over \mathcal{I} in (7.28) to include also the sets \mathcal{I} which do not contain 1 at a cost of an eligible error given by the bound (7.26) with $j^* = 1$. Finally, we conclude by observing that the analyticity of $E_1(s)$ for $(s, \alpha, \beta) \in \mathbb{C} \times \Omega_k$ (where Ω_k is defined in (6.3)) can be immediately verified from the definition of the various error terms.

Remark 9. Notice that the main term on the right of (7.28) is symmetric in N_1, \dots, N_k and, by the definition (7.20) of $\Psi_{\mathcal{I}, \varepsilon}$, with respect to $\mathbf{a} \leftrightarrow -\mathbf{a}$ (i.e. $\varepsilon \leftrightarrow -\varepsilon$).

8 Proof of Lemma 6

As in the proof of Lemma 5 we assume N_1 is the maximum among N_1, \dots, N_k . Writing the partitions of unity in terms of their Mellin transform, we obtain

$$A_{\alpha, \beta}(s) = \int_{(c_{u_1}, \dots, c_{u_k})} \mathcal{A}_{\alpha', \beta'}(s') \tilde{P}(u_1) \cdots \tilde{P}(u_k) N_1^{u_1} \cdots N_k^{u_k} du_1 \cdots du_k \quad (8.1)$$

where $s' := 1 + s - \sigma + \frac{\varepsilon}{k} - \eta'_{\alpha, \beta}$ with $\eta'_{\alpha, \beta} := \frac{1}{k} \sum_{i=1}^k \min(\Re(\alpha_i), \Re(\beta_i))$, $\alpha' := (-1 + \alpha_i + u_i - \frac{\varepsilon}{k} + \sigma + \eta'_{\alpha, \beta})_{i \in \{1, \dots, k\}}$ and $\beta' := (-1 + \beta_i + u_i - \frac{\varepsilon}{k} + \sigma + \eta'_{\alpha, \beta})_{i \in \{1, \dots, k\}}$. We move the lines of integration to

$$c_{u_i} = 1 - \sigma - \eta'_{\alpha, \beta} + \frac{\varepsilon}{k} \quad \forall i \in \{2, \dots, k\}, \quad c_{u_1} = -\sigma - \eta'_{\alpha, \beta} + \frac{\varepsilon}{k}$$

so that on the new lines of integration we have $|\Re(\alpha'_i)|, |\Re(\beta'_i)| \leq \frac{1}{2(k-1)}$ for $i = 2, \dots, k$ and

$$\Re(s') \geq 1 - \frac{1}{k} + \frac{\varepsilon}{k} - \frac{1}{k} \sum_{i=1}^k \min(\Re(\alpha'_i), \Re(\beta'_i)).$$

We cannot apply directly Lemma 4 since we would need also $|\Re(\alpha'_1)|, |\Re(\beta'_1)| \leq \frac{1}{2(k-1)}$, whereas we have $\Re(\alpha'_1) = -1 + \Re(\alpha_1)$, $\Re(\beta'_1) = -1 + \Re(\beta_1)$. However, we observe that

$$\begin{aligned} |\mathcal{A}_{\alpha, \beta}(s')| &\ll_{\varepsilon} \sum_{a_1 n_1 + \dots + a_k n_k = 0} \left| \frac{n_1^{-s'_1 + \varepsilon} d(n_2) \cdots d(n_k)}{n_2^{s'_2} \cdots n_k^{s'_k}} \right| \\ &\leq \left(1 + (k \max_{i=2}^k |a_i|)^{-s'_1 + \varepsilon}\right) \sum_{\ell=2}^k \sum_{\max_{j=2}^k n_j = n_{\ell}} \left| \frac{d(n_2) \cdots d(n_k)}{n_{\ell}^{s'_1 - \varepsilon} n_2^{s'_2} \cdots n_k^{s'_k}} \right| \end{aligned}$$

since $(\max_{j=2}^k n_j) \leq n_1 \leq k(\max_{j=2}^k n_j)(\max_{i=2}^k |a_i|)$ and where $s'_i = 1 + \frac{\varepsilon}{k} - \eta'_{\alpha, \beta} + \min(\Re(\alpha_i), \Re(\beta_i))$ for $i = 2, \dots, k$ and $s'_1 = \frac{\varepsilon}{k} - \eta'_{\alpha, \beta} + \min(\Re(\alpha_1), \Re(\beta_1))$. Then, we can proceed exactly as in the proof of Lemma 4 obtaining

$$|\mathcal{A}_{\alpha, \beta}(s')| \ll \max_{i=2}^k |a_i| \left(\frac{Ak}{\varepsilon} \right)^{4k}.$$

Thus, (8.1) gives

$$A_{\varepsilon; \alpha, \beta}(s) \ll_{\varepsilon} \max_{i=2}^k |a_i| \left(\frac{Ak}{\varepsilon} \right)^{4k} N_1^{-1+k\eta'_{\alpha, \beta}} (N_1 \cdots N_k)^{1-\sigma} \ll_{\varepsilon} X_{s, k, \alpha, \varepsilon}^{Ak} Y_{\alpha, \beta}^k (N_1 \cdots N_k)^{1-\sigma} N_{\max}^{-1+\eta_{\alpha, \beta} + \varepsilon}$$

since $k\eta'_{\alpha, \beta} \leq \eta_{\alpha, \beta}$ and the Lemma follows since by (7.25) and (7.26) the main term on the right hand side of (6.8) also satisfies the bound (6.9).

9 The completion of the proof of Theorem 3

We define

$$E^*(s) = \sum_{\substack{N_1, \dots, N_k, \\ B_1 < B_2}}^\dagger E_1(s) + \sum_{\substack{N_1, \dots, N_k, \\ B_1 > B_2}}^\dagger E_2(s)$$

with $E_1(s)$ and $E_2(s)$ as in Lemma 5 and 6, where

$$B_1 := \frac{N_{\max}^{\frac{k-1}{2}}}{(N_1 \cdots N_k)^\sigma} + \frac{N_{\max}^{\frac{k}{2}-1-\frac{1}{k-1}}}{(N_1 \cdots N_k)^{\sigma-\frac{1}{k-1}}} + \frac{(N_1 \cdots N_k)^{1-\frac{1}{k}-\sigma}}{N_{\max}} + N_{\max}^{\frac{k-1}{2}-\frac{k+1}{2}\sigma}$$

$$B_2 := (N_1 \cdots N_k)^{1-\frac{1}{k}-\sigma} N_{\max}^{-1}$$

Since $E_1(s)$ and $E_2(s)$ are entire for all N_1, \dots, N_k we have that $E^*(s)$ is holomorphic in the half-plane where the above sums converge absolutely (and uniformly in s). Thus, writing $N_1 \cdots N_k =: N_{\max}^x$ with $1 \leq x \leq k$, we see that $E^*(s)$ is holomorphic provided that $\frac{1}{2} \leq \sigma \leq 1$, $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega_k$ and

$$\max_{1 \leq x \leq k} \min_{i=1, \dots, 4} (\max(L_i(x)), (1 - \frac{1}{k} - \sigma)x - 1) + \eta_{\boldsymbol{\alpha}, \boldsymbol{\beta}} + 3\varepsilon \leq 0$$

where

$$L_1(x) := \frac{k-1}{2} - \sigma x, \quad L_2(x) := \frac{k}{2} - 1 - \frac{1}{k-1} - (\sigma - \frac{1}{k-1})x$$

$$L_3(x) := (1 - \frac{1}{k} - \sigma)x - 1, \quad L_4(x) := \frac{k-1}{2} - \frac{k+1}{2}\sigma.$$

Equivalently, we need

$$M_i := \max_{1 \leq x \leq k} \min(L_i(x), (1 - \sigma)x - 1) + \eta_{\boldsymbol{\alpha}, \boldsymbol{\beta}} + 3\varepsilon < 0$$

for $i = 1, \dots, 4$. For $\frac{1}{2} \leq \sigma \leq 1$ one has

$$M_1 = M_2 = M_4 = (1 - \sigma)\frac{k+1}{2} - 1, \quad M_3 = \max(k - 2 - k\sigma, -\frac{1}{k} - \sigma).$$

Thus, we need $1 - \frac{2(1-\eta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}-3\varepsilon)}{k+1} \leq \sigma \leq 1$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega_k$. Also, in this strip we have the bound

$$E^*(s) \ll_\varepsilon X_{s,k,\boldsymbol{\alpha},\varepsilon}^{Ak} Y_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{7k}. \quad (9.1)$$

Finally, for any function $F(u)$ which is analytic and grows at most polynomially on a strip $|\Re(u)| < c$ for some $c > 0$ we have

$$\sum_N^\dagger \frac{1}{2\pi i} \int_{(0)} F(u) \tilde{P}(u) N^u du = F(0).$$

Thus, by the definition (7.24) of $Z_{\mathcal{I};\alpha,\beta}(s)$, for

$$1 - \frac{1}{k} + \frac{\varepsilon}{k} - \frac{1}{k} \sum_{i=1}^k \min(\Re(\alpha_i), \Re(\beta_i)) \leq \sigma < 1 - \max_{1 \leq i \leq k} \max(\Re(\alpha_i), \Re(\beta_i)), \quad (\alpha, \beta) \in \Omega_k \quad (9.2)$$

we have

$$\sum_{N_1, \dots, N_k}^{\dagger} \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, \\ |\mathcal{I}| > \frac{k+1}{2}}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i, \beta_i\} \quad \forall i \in \mathcal{I}, \\ (\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j) \quad \forall j \in \mathcal{J}}} \left(\prod_{i \in \mathcal{I}} \zeta(1 - \alpha_i + \beta_i) \right) Z_{\mathcal{I};\alpha^*,\beta^*}(s) = \mathcal{W}_{\alpha,\beta}(s),$$

where $\mathcal{W}_{\alpha,\beta}(s)$ is defined as

$$\begin{aligned} \mathcal{W}_{\alpha,\beta}(s) := & \sum_{\substack{\mathcal{I} \cup \mathcal{J} = \{1, \dots, k\}, \\ \mathcal{I} \cap \mathcal{J} = \emptyset, \\ |\mathcal{I}| > \frac{k+1}{2}}} \sum_{\substack{\{\alpha_i^*, \beta_i^*\} = \{\alpha_i, \beta_i\} \quad \forall i \in \mathcal{I}, \\ (\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j) \quad \forall j \in \mathcal{J}}} \Psi_{\varepsilon, \mathcal{I}}(\mathbf{v}'_{\mathcal{I}}) \left(\prod_{i \in \mathcal{I}} \zeta(1 - \alpha_i^* + \beta_i^*) |a_i|^{-1 + \alpha_i^* + s} \right) \\ & \times \frac{G(1 - \sum_{i \in \mathcal{I}} (1 - \alpha_i^* - s))}{1 - \sum_{i \in \mathcal{I}} (1 - \alpha_i^* - s)} \sum_{\ell \geq 1} \frac{\prod_{i \in \mathcal{I}} (a_i, \ell)^{1 - \alpha_i^* + \beta_i^*}}{\ell^{\sum_{i \in \mathcal{I}} (1 - \alpha_i^* + \beta_i^*)}} \sum_{h \pmod{\ell}}^* \prod_{j \in \mathcal{J}} D_{\alpha_j^*, \beta_j^*}(s, \frac{ha_j}{\ell}) \end{aligned}$$

where $\mathbf{v}'_{\mathcal{I}} = (1 - \alpha_i - s)_{i \in \mathcal{I}}$ (notice that the inequality (9.2) ensures the convergence of the series over ℓ).

We notice that we can assume that

$$1 - \frac{1}{k} + \frac{\varepsilon}{k} - \frac{1}{k} \sum_{i=1}^k \min(\Re(\alpha_i), \Re(\beta_i)) < 1 - \frac{\varepsilon}{k} - \max_{1 \leq i \leq k} \max(\Re(\alpha_i), \Re(\beta_i)) \quad (9.3)$$

so that the strip for s in (9.2) is non-empty. Indeed, if (9.3) doesn't hold then we would have

$$- \sum_{i=1}^k \min(\Re(\alpha_i), \Re(\beta_i)) > 1 - \varepsilon - k \max_{1 \leq i \leq k} \max(\Re(\alpha_i), \Re(\beta_i))$$

and so $\frac{2}{3}\eta_{\alpha,\beta} > 1 - \varepsilon - k \max_{1 \leq i \leq k} |\max(\Re(\alpha_i), \Re(\beta_i))|$ which implies

$$\begin{aligned} (2k-1) \sum_{i=1}^k (|\Re(\alpha_i)| + |\Re(\beta_i)|) &> (2k-1)(1-2\varepsilon) - (2k^2-k) \max_{1 \leq i \leq k} |\max(\Re(\alpha_i), \Re(\beta_i))| \\ &> k-1 - \frac{1}{2(k-1)} - \varepsilon(4k-2) - (k+1) \max_{1 \leq i \leq k} |\max(\Re(\alpha_i), \Re(\beta_i))|, \end{aligned}$$

since $\max_{1 \leq i \leq k} |\max(\Re(\alpha_i), \Re(\beta_i))| \leq \frac{1}{2(k-1)}$. Thus, one also has

$$(2k-1) \sum_{i=1}^k (|\Re(\alpha_i)| + |\Re(\beta_i)|) + (k-1) \sum_{i=1}^k |\max(\Re(\alpha_i), \Re(\beta_i))| > k-1 - \frac{1}{2(k-1)} - \varepsilon(4k-2)$$

and so

$$2k\eta_{\alpha,\beta} + (k+1) \sum_{i=1}^k \min(\Re(\alpha_i), \Re(\beta_i)) > k - 1 - \frac{1}{2(k-1)} - \varepsilon(4k-2).$$

In particular, choosing suitably the values of ε in the two statements, Theorem 3 is implied by Lemma 4 in this case.

Now, by the definition of $A_{\alpha,\beta}(s)$, if $\sigma \leq 1 - \max_{1 \leq i \leq k} \max(\Re(\alpha_i), \Re(\beta_i))$ we have

$$\sum_{N_1, \dots, N_k}^{\dagger} A_{\alpha,\beta}(s) = \mathcal{A}_{\mathbf{a};\alpha,\beta}(s),$$

whence we have $\mathcal{A}_{\mathbf{a};\alpha,\beta}(s) = \mathcal{W}_{\alpha,\beta}(s) + E^*(s)$ for (s, α, β) satisfying (9.2). Then, we observe that, since $G(0) = 1$, one has that $\mathcal{W}_{\alpha,\beta}(s) - \mathcal{M}_{\mathbf{a};\alpha,\beta}(s)$ is holomorphic for $\frac{1}{2} \leq \sigma \leq 1 - \max_{1 \leq i \leq k} \max(\Re(\alpha_i), \Re(\beta_i)) - \frac{\varepsilon}{k}$ and $(\alpha, \beta) \in \Omega_k$ (with $\mathcal{M}_{\mathbf{a};\alpha,\beta}(s)$ defined in (2.3)), and one easily checks that in this region $\mathcal{W}_{\alpha,\beta}(s) - \mathcal{M}_{\mathbf{a};\alpha,\beta}(s)$ satisfies the same bound as in (9.1). Thus, taking

$$E(s) := E^*(s) + \mathcal{W}_{\alpha,\beta}(s) - \mathcal{M}_{\mathbf{a};\alpha,\beta}(s)$$

we obtain Theorem 3 for

$$1 - \frac{2(1-\eta_{\alpha,\beta}-3\varepsilon)}{k+1} \leq \sigma \leq 1 - \max_{1 \leq i \leq k} \max(\Re(\alpha_i), \Re(\beta_i)) - \frac{\varepsilon}{k}.$$

Finally, by (9.3) one has that for $\sigma \geq 1 - \max_{1 \leq i \leq k} \max(\Re(\alpha_i), \Re(\beta_i)) - \frac{\varepsilon}{k}$ Theorem 3 is a consequence of Lemma 4 and so the proof of Theorem 3 is complete.

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