ON THE DISTRIBUTION OF A COTANGENT SUM

SANDRO BETTIN

Abstract. Maier and Rassias computed the moments and proved a distribution result for the cotangent sum $\mathrm{c}_0(a/q) := -\sum_{m<q} \frac{m}{q}\cot(\pi ma/q)$ on average over $1/2 < A_0 \leq a/q < A_1 < 1$, as $q \to \infty$. We give a simple argument that recovers their results (with stronger error terms) and extends them to the full range $1 \leq a < q$. Moreover, we give a density result for c_0 and answer a question posed by Maier and Rassias on the growth of the moments of c_0.

1. Introduction

In this note, we consider the cotangent sum

$$\mathrm{c}_0(a/q) := -\sum_{m=1}^{q-1} \frac{m}{q}\cot\Big(\frac{\pi ma}{q}\Big), \qquad (a,q)=1, q\geq 1,$$

which is related to the Báez-Duarte-Nyman-Beurling criterion for the Riemann hypothesis (see, for example, [Bag, BC]) and was recently studied in [BC] and [MR]. In [BC], Conrey and the author proved that c_0 satisfies the reciprocity formula

$$\mathrm{c}_0(a/q) + q/a\,\mathrm{c}_0(q/a) - (\pi q)^{-1} = \psi(a/q), \tag{1.1}$$

where $\psi(x)$ is an analytic function in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ satisfying several nice properties. In particular, it satisfies the three term relation $\psi(x) = \psi(x+1) + (x+1)^{-1}\psi(x/(x+1))$ and has an asymptotic expansion for $x \to 0$, starting by

$$\psi(x) = -\frac{\log(2\pi x) - \gamma}{\pi x} + O(\log x). \tag{1.2}$$

Ishibashi [Ish] observed that c_0 is also related to the value at $s=0$, or $s=1$ by the functional equation, of the ("imaginary part" of the) Estermann function:

$$\mathrm{c}_0(a/q) = 2D_{\sin}(0,a/q) = 2q\pi^{-2}D_{\sin}(1,\overline{a}/q), \tag{1.3}$$

where for $x\in\mathbb{R}$, $\Re(s)>1$,

$$D_{\sin}(s,x) := \sum_{n=1}^{\infty} \frac{d(n)\sin(2\pi nx)}{n^s}$$

with $d(n)$ indicating the divisor function. If $x\in\mathbb{Q}$, then $D_{\sin}(s,x)$ can be extended to an entire function of s satisfying the functional equation

$$\Lambda_{\sin}(s,a/q) := \Gamma\big(\tfrac{1+s}{2}\big)^2 (q/\pi)^s D_{\sin}(s,a/q) = \Lambda_{\sin}(1-s,\overline{a}/q), \tag{1.4}$$

where $\overline{a}$ denotes the inverse of a modulo the denominator q.

If $x \in \mathbb{R} \setminus \mathbb{Q}$, then Wilton [Wil] proved that the convergence of the series at $s = 1$ is equivalent to the convergence of $\sum_{n \geq 1} (-1)^n \log v_{n+1}/v_n$, where u_n/v_n denotes the n-th partial quotient of x. Moreover, de la Bretèche and Tenenbaum [dBT] showed that for $x \notin \mathbb{Q}$

$$D_{\sin}(1, x) := \sum_{n=1}^{\infty} \frac{d(n) \sin(2\pi n x)}{n} = \pi \sum_{n=1}^{\infty} \frac{\frac{1}{2} - \{nx\}}{n}$$

whenever one of the two series converges, where $\{x\}$ denotes the fractional part of x.

Recently, Maier and Rassias [MR] computed the moments of $\mathrm{c}_0(a/q)$ proving that

$$\frac{1}{(A_1 - A_0)\varphi(q)} \sum_{\substack{(a,q)=1,\\ A_0 < a/q < A_1}} \mathrm{c}_0(a/q)^k = H_k q^k (1 + o(1))$$

as $q \to \infty$, for certain constants H_k and any fixed $\frac{1}{2} < A_0 < A_1 < 1$ (or, equivalently, $0 < A_0 < A_1 < \frac{1}{2}$, since c_0 is odd and 1-periodic) and where $\phi(q)$ is Euler's function. They also computed the distribution of $\frac{1}{q}\,\mathrm{c}_0(a/q)$ and proved that

$$\frac{1}{(A_1 - A_0)\varphi(q)} \sum_{\substack{(a,q)=1,\\ A_0 < a/q < A_1}} f\Big(\tfrac{1}{q}\,\mathrm{c}_0(a/q)\Big) = (1 + o(1)) \int_{\mathbb{R}} f(x) dF(x), \tag{1.5}$$

as $q \to \infty$ for any continuous function of compact support $f(x)$ and where $F(x)$ is the continuous (as it is shown in the same paper) function defined by

$$F(x) := \mathrm{meas}\big(\{z \in [0,1] \mid 2\pi^{-2} D_{\sin}(1, z) \leq x\}\big).$$

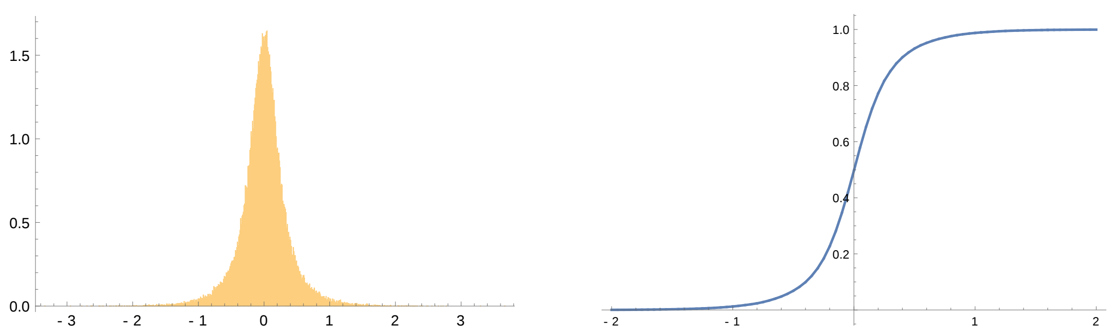

FIGURE 1. The histogram of $2\pi^{-2} D_{\sin}(1, x)$ and the graph of $F(x)$ obtained by sampling $D_{\sin}(1, x)$ (truncated at $n \leq 10^5$) at 10^5 points chosen uniformly in $[0, 1]$.

In this paper we strengthen the results of [MR] and extend them to any $0 \leq A_0 < A_1 \leq 1$, including the full range $A_0 = 0, A_1 = 1$.

Theorem 1. *Let $q \geq 1$ and $k \geq 0$. Then,*

$$\frac{1}{\varphi(q)} \sum_{\substack{a=1,\\(a,q)=1}}^{q} \mathrm{c}_0(a/q)^k = H_k q^k + O_\varepsilon(q^{k-1+\varepsilon}(Ak\log q)^{2k}), \tag{1.6}$$

for some absolute constant $A > 0$ and any $\varepsilon > 0$, where

$$H_k := (i\pi^2)^{-k} \sum_{\substack{(n_1,\ldots,n_k)\in(\mathbb{Z}_{\neq 0})^k,\\ n_1+\cdots+n_k=0}} \frac{d(|n_1|)\cdots d(|n_k|)}{n_1\cdots n_k}.$$

Furthermore, if $0 \leq A_0 < A_1 \leq 1$, then we have

$$\frac{1}{\varphi(q)} \sum_{\substack{(a,q)=1,\\ A_0<a/q<A_1}} \mathrm{c}_0(a/q)^k = (A_1 - A_0)H_k q^k + O_\varepsilon(q^{k-\frac{1}{2}+\varepsilon}(Ak\log q)^{2k}). \tag{1.7}$$

Remark 1. *If k is odd, then both H_k and the left hand side of* (1.6) *are identically zero.*

In the same paper Maier and Rassias ask whether $\sum_{k=0}^{\infty} \frac{H_k t^k}{k!}$ has positive radius of convergence. The following Theorem answers their question in the affirmative.

Theorem 2. *We have $H_k \ll A^k k!$ for some $A > 0$.*

The fact that the series $\sum_{k=0}^{\infty} H_k t^k/k!$ has a positive radius of convergence guarantees that the moments H_k univocally determine the distribution function of c_0. Thus, we can deduce the following Corollary.

Corollary 1. *Let $q \geq 1$ and let $f(x)$ be a continuous function with compact support. Let $0 \leq A_0 < A_1 \leq 1$ with $A_1 - A_0 \gg q^{-\frac{1}{2}+\delta}$ for some fixed $\delta > 0$. Then, as $q \to \infty$ we have*

$$\frac{1}{(A_1 - A_0)\varphi(q)} \sum_{\substack{a=1,\\(a,q)=1}}^{q} f\left(\tfrac{1}{q}\,\mathrm{c}_0\left(\tfrac{a}{q}\right)\right) = (1 + o(1)) \int_{\mathbb{R}} f(x)dF(x).$$

Remark 2. *Notice that by* (1.3) *Theorem 1 and 2 automatically give the analogous results for $D_{\sin}(1, a/q)$ (or, equivalently, for the Vasyunin sum $V(a,q) := -\mathrm{c}_0(\overline{a}/q)$) when $A_0 = 0$ and $A_1 = 1$, as can be seen by making the change of variable $a \to \overline{a}$.*

Remark 3. *It is not necessary to compute and bound all the moments of c_0 to prove Corollary 1. Indeed, one can take the alternative route of computing the distribution of $D_X(1,x)$, as defined in* (3.2) *below, for any fixed $X \geq 1$ by appealing to Theorem 4 of* [DM]*. From this (together with bounds for the second moments of $D_X(1,x)$ and of $D_{\sin}(1,x) - D_X(1,x)$), one can then deduce Corollary 1 by following the same approach of Section 6 of* [DM] *.*

Using (1.1), we can also give an alternative expression for $D_{\sin}(1,x)$ in terms of the denominators of the partial quotient of x.

Proposition 1. *Let $\langle a_0; a_1, a_2, \dots\rangle$ be the continued fraction expansion of $x \in \mathbb{R}$. Moreover, let u_r/v_r be the r-th partial quotient of x. Then*

$$D_{\sin}(1,x) := \sum_{n=1}^{\infty} \frac{d(n)\sin(2\pi n x)}{n} = -\frac{\pi^2}{2}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{v_\ell}\left(\frac{1}{\pi v_\ell} + \psi\left(\frac{v_{\ell-1}}{v_\ell}\right)\right) \tag{1.8}$$

whenever either of the two series is convergent.

If $x = \langle a_0; a_1, \dots, a_r\rangle$ is a rational number, then the range of summation of the series on the right is to be interpreted to be $1 \le \ell \le r$.

Remark 4. *If $x \in \mathbb{Q}$ then one can write two different continued fraction expansion for x, but (1.8) holds regardless of the chosen expansion.*

Proposition 1, which constitutes a refinement of the aforementioned result of Wilton, can be interpreted as an extension of the reciprocity formula (1.1) to $x \notin \mathbb{Q}$ (see also Theorem 2 of [BM], where another formula of similar flavor for $D_{\sin}(1,x)$ is given). We also remark that Proposition 1 is of independent interest as $D_{\sin}(1,a/q) = -\frac{\pi^2}{2q}V(a,q)$ is exactly the sum appearing in the Báez-Duarte criterion for the Riemann hypothesis (c.f. [BC]).

The exact formula (1.8) allows us to prove the following corollary which, combined with (1.5) and the periodicity modulo 1 of c_0, implies that $\{(a/q, \frac{1}{q}c_0(a/q)) \mid (a,q) = 1, q \ge 1\}$ is dense in $\mathbb{R}^2$.

Corollary 2. *The function $F(x)$ is strictly increasing.*

Corollary 3. *We have that $\{(a/q, \frac{1}{q}c_0(a/q)) \mid (a,q) = 1, q \ge 1\}$ is dense in $\mathbb{R}^2$.*

2. Proof of Theorem 1

Both in this and in the following section, we will often consider $D_{\sin}(1,a/q)$ rather than $c_0(a/q)$. The stated result then follows by (1.3). Moreover, we assume that k is even, as the result is trivial otherwise.

Let $g(x)$ be a smooth function such that $g(x) = 1$ for $0 \le x \le 1$, $0 \le g(x) \le 1$ for $1 \le x \le 2$ and $g(x) = 0$ for $x \ge 2$. The Mellin transform of g satisfies

$$\hat{g}(s) := \int_0^\infty g(x)x^{s-1}\,dx = \frac{1}{s} + \int_1^2 g(x)x^{s-1}\,dx$$

for $\Re(s) > 0$ and thus extends to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 0$ with residue 1. Moreover, by repeated integration by parts one sees that $\hat{g}(s) \ll_j |s|^{-j}$ for any $j \ge 0$ and $\Re(s) \ll 1$. Thus, by the residue theorem we have

$$\begin{aligned}\sum_{n\le 2X} g(n/X)\frac{d(n)\sin(2\pi n a/q)}{n} &= \int_{(1)} \hat{g}(s)D_{\sin}(1+s,a/q)X^s\,ds \\ &= D_{\sin}(1,a/q) + \int_{(-1-\varepsilon_1)} \hat{g}(s)D_{\sin}(1+s,\tfrac{a}{q})X^s ds\end{aligned}$$

for $\varepsilon_1 = 0.1$ and any $X \geq 1$. Bounding trivially the integral on the last line (using the functional equation (1.4)), we obtain

$$D_{\sin}(1, a/q) = \sum_{n \leq 2X} g(n/X) \frac{d(n)\sin(2\pi n a/q)}{n} + O(q^{1+2\varepsilon_1}/X).$$

Assuming $X \geq q^{1+2\varepsilon_1}$ and using Euler's formula, we can thus express $D_{\sin}(1, a/q)^k$ as

$$\begin{aligned} D_{\sin}(1, a/q)^k = (2i)^{-k} \sum_{n_1,\dots,n_k \in B^*_{2X}} \mathrm{e}\Big((n_1 + \cdots + n_k)\frac{a}{q}\Big) \frac{\tilde{d}(n_1,\dots,n_k)}{n_1 \cdots n_k} + \\ + O\big((A \log X)^{2k} X^{-1} q^{1+2\varepsilon_1}\big), \end{aligned} \tag{2.1}$$

where $B^*_X := [-X, X] \cap \mathbb{Z}_{\neq 0}$,

$$\tilde{d}(n_1,\dots,n_k) := d(|n_1|) \cdots d(|n_k|) g(|n_1|/X) \cdots g(|n_k|/X)$$

and A denotes an absolute positive constant that might change from line to line. Thus, by the Möbius inversion formula and the orthogonality of additive characters we have

$$\begin{aligned} \frac{1}{\varphi(q)} \sum_{\substack{a=1,\\ (a,q)=1}}^{q} D_{\sin}(1, a/q)^k &= \sum_{\ell | q} \frac{\mu(q/\ell)}{\varphi(q)} \sum_{a=1}^{\ell} D_{\sin}(1, a/\ell)^k \\ &= (2i)^{-k} \sum_{\ell | q} \frac{\mu(q/\ell)\ell}{\varphi(q)} \sum_{\substack{n_1,\dots,n_k \in B^*_{2X},\\ n_1+\cdots+n_k \equiv 0 \pmod{\ell}}} \frac{\tilde{d}(n_1,\dots,n_k)}{n_1 \cdots n_k} + O\bigg(\frac{(A\log X)^{2k} q^{1+3\varepsilon_1}}{X}\bigg). \end{aligned}$$

The contribution of the terms with $n_1 + \cdots + n_k \neq 0$ is bounded by

$$\begin{aligned} &\sum_{\ell | q} \frac{\ell}{\varphi(q)} \sum_{\substack{n_1,\dots,n_k \in B^*_{2X},\\ 0 \neq n_1+\cdots+n_k \equiv 0 \pmod{\ell}}} \frac{\tilde{d}(n_1,\dots,n_k)}{|n_1 \cdots n_k|} \\ &\ll_\varepsilon \sum_{\ell | q} \frac{k\ell}{\varphi(q)} \sum_{\substack{n_1,\dots,n_k \in B^*_{2X},\\ |n_1| \geq \ell/k,\\ 0 \neq n_1+\cdots+n_k \equiv 0 \pmod{\ell}}} \frac{X^\varepsilon d(|n_2|) \cdots d(|n_k|)}{|n_1 \cdots n_k|} \ll_\varepsilon q^{-1+\varepsilon} A^k X^\varepsilon (\log X)^{2k}, \end{aligned} \tag{2.2}$$

since

$$\sum_{\substack{\ell/k \leq n \leq 2X,\\ n \equiv c \pmod{\ell}}} \frac{1}{n} \ll \frac{k}{\ell} + \frac{\log X}{\ell}.$$

The contribution of the diagonal term $n_1 + \cdots + n_k = 0$ is

$$\begin{aligned} \sum_{\ell | q} \frac{\mu(q/\ell)\ell}{\varphi(q)} (2i)^{-k} \sum_{\substack{n_1,\dots,n_k \in B^*_{2X},\\ n_1+\cdots+n_k = 0}} \frac{\tilde{d}(n_1,\dots,n_k)}{|n_1 \cdots n_k|} &= \frac{\pi^{2k}}{2^k} H_k \sum_{\ell | q} \frac{\mu(q/\ell)\ell}{\varphi(q)} + O_\varepsilon((Ak)^{2k} q^\varepsilon X^{-\frac{1}{2}+\varepsilon}) \\ &= \frac{\pi^{2k}}{2^k} H_k + O_\varepsilon((Ak)^{2k} q^\varepsilon X^{-\frac{1}{2}+\varepsilon}), \end{aligned}$$

since

$$\sum_{\substack{(n_1,\dots,n_k)\in(\mathbb{Z}_{\neq0})^k\setminus(B_{2X}^*)^k,\\ n_1+\cdots+n_k=0}} \frac{\tilde{d}(n_1,\dots,n_k)}{|n_1\cdots n_k|} \ll \frac{k}{X^{\frac12-\varepsilon}} \sum_{n_2,\dots,n_k\in\mathbb{Z}_{\neq0}} \frac{d(|n_2|)\cdots d(|n_k|)}{|n_2\cdots n_k|^{1+\frac{1/2}{k-1}}} \ll (Ak)^{2k} X^{-\frac12+\varepsilon}, \tag{2.3}$$

where we used $|n_1|\geq X^{\frac12}|n_2\cdots n_k|^{\frac{1}{2(k-1)}}$ if $|n_2|,\dots,|n_k|,X\leq|n_1|$. Equation (1.6) then follows upon choosing $X=q^3$.

The proof of (1.7) is very similar. Indeed, by (1.3) one has

$$\frac{\pi^{2k}}{(2q)^k\varphi(q)} \sum_{\substack{(a,q)=1,\\ A_0<a/q<A_1}} \mathrm{c}_0(a/q)^k = \frac{1}{\varphi(q)} \sum_{\substack{(a,q)=1,\\ A_0<a/q<A_1}} D(1,\overline{a}/q)^k$$

and so by (2.1) we have that this is equal to

$$\frac{(2i)^{-k}}{\varphi(q)} \sum_{n_1,\dots,n_k\in B_{2X}^*} \frac{\tilde{d}(n_1,\dots,n_k)}{n_1\cdots n_k} \sum_{\substack{(a,q)=1,\\ A_0<a/q<A_1}} \mathrm{e}\bigg((n_1+\cdots+n_k)\frac{\overline{a}}{q}\bigg) + O\big((A\log X)^{2k}X^{-1}q^{1+2\varepsilon_1}\big).$$

By the Weil bound for incomplete Kloosterman sums (as in Lemma 8 of [DFI]), the contribution of the terms with $n_1+\cdots+n_k\neq0$ is

$$\ll_\varepsilon \frac{q^{\frac12+\varepsilon}}{\varphi(q)} \sum_{\substack{n_1,\dots,n_k\in B_{2X}^*,\\ n_1+\cdots+n_k\neq0}} \frac{\tilde{d}(n_1,\dots,n_k)}{|n_1\cdots n_k|}(n_1+\cdots+n_k,q)$$

$$\ll_\varepsilon \sum_{\ell|q} \frac{q^{\frac12+\varepsilon}\ell}{\varphi(q)} \sum_{\substack{n_1,\dots,n_k\in B_{2X}^*,\\ 0\neq n_1+\cdots+n_k\equiv0 \pmod{\ell}}} \frac{\tilde{d}(n_1,\dots,n_k)}{|n_1\cdots n_k|} \ll_\varepsilon q^{-\frac12+\varepsilon}A^kX^\varepsilon(\log X)^{2k},$$

by (2.2). The contribution of the terms with $n_1+\cdots+n_k=0$ is

$$\frac{(2i)^{-k}}{\varphi(q)} \sum_{\substack{n_1,\dots,n_k\in B_{2X}^*,\\ n_1+\cdots+n_k=0}} \frac{\tilde{d}(n_1,\dots,n_k)}{n_1\cdots n_k} \sum_{\substack{(a,q)=1,\\ A_0<a/q<A_1}} 1$$

$$= (2i)^{-k} \sum_{\substack{n_1,\dots,n_k\in B_{2X}^*,\\ n_1+\cdots+n_k=0}} \frac{\tilde{d}(n_1,\dots,n_k)}{n_1\cdots n_k}((A_1-A_0)+O(q^{-1+\varepsilon}))$$

$$= \frac{\pi^{2k}}{2^k}(A_1-A_0)H_k + O_\varepsilon((Ak)^{2k}q^\varepsilon X^{-\frac12+\varepsilon} + (A\log X)^{2k}q^{-1+\varepsilon}))$$

by (2.3) and since by the Mobiüs inversion formula we have

$$\sum_{\substack{(a,q)=1,\\ A_0<a/q<A_1}} 1 = \sum_{d|q}\mu(d) \sum_{\substack{a\equiv0 \pmod{d},\\ A_0<a/q<A_1}} 1 = \sum_{d|q}\mu(d)((A_1-A_0)\frac{q}{d}+O(1))$$

$$= (A_1-A_0)\varphi(q) + O(d(q)).$$

As before, we take $X = q^3$ and (1.7) follows.

3. Proof of Proposition 1 and Theorem 2

The following lemmas give Proposition 1 in the cases of $x \in \mathbb{Q}$ and $x \notin \mathbb{Q}$ respectively.

Lemma 4. *Let $(a,q)=1$, $q>0$ and let $v_0,\dots,v_r$ be the partial denominators of the continued fraction expansion $a/q=\langle a_0; a_1,\dots,a_r\rangle$. Then*

$$D_{\sin}(1,a/q) = -\frac{\pi^2}{2}\sum_{\ell=1}^{r}\frac{(-1)^\ell}{v_\ell}\bigg(\frac{1}{\pi v_\ell}+\psi\Big(\frac{v_{\ell-1}}{v_\ell}\Big)\bigg).$$

Proof. Write $\frac{b}{q} := (-1)^{r+1}\frac{\overline{a}}{q}$. Then, one has the continued fraction expansion $b/q = \langle 0; b_1,\dots,b_r\rangle = \langle 0; a_r,\dots,a_1\rangle$. Moreover, the Euclid algorithm for b/q gives

$$\begin{aligned} &y_1 = q, \qquad y_2 = b,\\ &y_{n-1} = b_{n-1}y_n + y_{n+1}, \qquad n = 1,\dots r+1, \end{aligned}$$

with $y_{r+1-\ell} = v_\ell$, where v_ℓ is the ℓ-th partial quotient of a/q (as usual we put $v_{-1} := 0$). Thus, applying repeatedly the reciprocity formula (1.1) and using that $c_0(1)=0$, we obtain

$$\frac{1}{q}c_0(b/q) = -\sum_{m=1}^{r}\frac{(-1)^m}{\pi y_m^2} - \sum_{m=1}^{r}\frac{(-1)^m}{y_m}\psi\Big(\frac{y_{m+1}}{y_m}\Big) = \sum_{\ell=1}^{r}\frac{(-1)^{\ell+r}}{v_\ell}\bigg(\frac{1}{\pi v_\ell}+\psi\Big(\frac{v_{\ell-1}}{v_\ell}\Big)\bigg)$$

and the Lemma follows by (1.3) since $D_{\sin}(s,x) = -D_{\sin}(s,-x)$. □

Lemma 5. *Let $x\in\mathbb{R}\backslash\mathbb{Q}$ and assume x has continued fraction expansion $x=\langle a_0;a_1,a_2,\dots\rangle$ with partial denominators $v_0,v_1,v_2,\dots$ Then*

$$D_{\sin}(1,a/q) = -\frac{\pi^2}{2}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{v_\ell}\bigg(\frac{1}{\pi v_\ell}+\psi\Big(\frac{v_{\ell-1}}{v_\ell}\Big)\bigg), \tag{3.1}$$

whenever $D_{\sin}(1,a/q)$ is defined. Moreover, writing

$$D_X(1,x) := \sum_{n\le X}\frac{d(n)\sin(2\pi nx)}{n}, \qquad S(x) := \sum_{n=1}^{\infty}\frac{\log v_{n+1}}{v_n}, \tag{3.2}$$

we have $D_X(1,x) \ll S(x)$ and $D_{\sin}(1,x)\ll S(x)$, uniformly in $x\in[0,1]\setminus\mathbb{Q}$, $X\ge 2$.

Proof. For a large positive constant $B\ge 5$, let $\xi_r = v_r(\log v_r)^B$ and let R be the maximum integer such that $\xi_R\le X$. We can split $D_X(1,x)$ into

$$D_X(1,x) = \sum_{n\le\xi_R} d(n)\frac{\sin(2\pi nx)}{n} + \sum_{\xi_R\le n\le X} d(n)\frac{\sin(2\pi nx)}{n}. \tag{3.3}$$

The second addend can be treated using the work of de la Bretèche and Tenenbaum [dBT]. Indeed, by partial summation, if B is sufficiently large we have

$$\sum_{\xi_R\leq n\leq X} d(n)\frac{\sin(2\pi nx)}{n} = \sum_{\xi_R\leq n\leq X} d(n)\frac{\sin(2\pi nx)}{X} + \int_{\xi_R}^{X}\sum_{\xi_R\leq n\leq t} d(n)\frac{\sin(2\pi nx)}{t^2}\,dt$$
$$= O\bigg(\frac{\log(v_{R+1})}{v_R}+\frac{1}{\log(v_R)}\bigg)$$

by (11.1) and (11.4) of [dBT]. For the first addend of the right hand side of (3.3), we first observe that

$$\sum_{n\leq\xi_R}\frac{d(n)\sin(2\pi nx)}{n} = \sum_{n\leq\xi_R}\frac{d(n)\sin(2\pi n\frac{u_R}{v_R})}{n} + O\bigg(\frac{(\log v_R)^{1+B}}{v_{R+1}}\bigg)$$

since $|x-u_R/v_R|\leq (v_Rv_{R+1})^{-1}$. Moreover, we observe that by Mellin's formula we have

$$\sum_{n\leq\xi_R} d(n)\frac{\sin(2\pi n\frac{u_R}{v_R})}{n} = D_{\sin}(1,u_R/v_R) + \frac{1}{2\pi i}\int_C D_{\sin}(1+s,u_R/v_R)\xi_R^s\frac{ds}{s} + O((\log\xi_R)^2/T)$$
$$= D_{\sin}(1,u_R/v_R) + O(v_RT^{\frac12}(\log v_R)^2/\xi_R + (\log v_R)^2/T)$$
$$= D_{\sin}(1,u_R/v_R) + O((\log v_R)^{-1}),$$

where C denotes the line from $s=(-1-\frac{1}{\log x})-iT$ to $s=(-1-\frac{1}{\log x})+iT$ with $T=(\log v_R)^4$ and where to bound the integral we used the functional equation (1.4) and a trivial bound.

Finally, by Lemma 4 we have

$$D_{\sin}(1,u_R/v_R) = -\frac{\pi^2}{2}\sum_{\ell=1}^{R}\frac{(-1)^\ell}{v_\ell}\bigg(\frac{1}{\pi v_\ell}+\psi\bigg(\frac{v_{\ell-1}}{v_\ell}\bigg)\bigg)$$

and thus

$$D_X(1,x) = -\frac{\pi^2}{2}\sum_{\ell=1}^{R}\frac{(-1)^\ell}{v_\ell}\bigg(\frac{1}{\pi v_\ell}+\psi\bigg(\frac{v_{\ell-1}}{v_\ell}\bigg)\bigg) + O\bigg(\frac{\log(v_{R+1})}{v_R}+\frac{1}{\log(v_R)}\bigg).$$

As $X\to\infty$, we have $v_R\to\infty$ and, by Theorem 4.4 of [dBT], $\log(v_{R+1})/v_R\to 0$ if (and only if) the series defining $S(1,x)$ converges and thus we obtain (3.1). The second assertion of the Lemma then follows by (1.2). □

We need two results from Khinchin's book on continued fractions [Khi].

Lemma 6. *For all $x\in\mathbb{R}\setminus\mathbb{Q}$ and all $n\geq 1$ we have $v_n\geq 2^{\frac{n-3}{2}}$.*

Proof. This is Theorem 12 of [Khi]. □

The following lemma is a minor refinement of Theorem 31 of [Khi].

Lemma 7. *Let $K\geq 1$. Then for all $\varepsilon>0$, there exists $B_\varepsilon>0$ such that*

$$E(K) := \mathrm{meas}\big(\{x\in[0,1]\mid v_r(x)\geq Ke^{B_\varepsilon r}\text{ for some } r\geq 1\}\big) \ll_\varepsilon K^{-1+\varepsilon}.$$

Proof. Proceeding as in the proof of Theorem 31 of [Khi], we see that for all $n, B \geq 1$ we have

$$\operatorname{meas}(E_n(K)) \ll \frac{2^n}{Ke^{Bn}} \sum_{\ell=0}^{n-1} \frac{(\log(Ke^{Bn}))^\ell}{\ell!},$$

where $E_n(K) := \left(\left\{x \in [0,1] \mid v_n(x) \geq Ke^{Bn}\right\}\right)$ (this is the first equation on page 68 of [Khi], with $g = Ke^n$). Now, if $K \leq e^{Bn}$ and B is large enough, then

$$\operatorname{meas}(E_n(K)) \ll \frac{2^n}{Ke^{Bn}} \sum_{\ell=0}^{n-1} \frac{(\log(Ke^{Bn}))^\ell}{\ell!} \ll \frac{2^n}{Ke^{Bn}} \sum_{\ell=0}^{n-1} \frac{(2Bn)^\ell}{\ell!} \ll \frac{n}{K} \frac{(4ne^{-B})^n}{n!} \ll \frac{e^{-Bn/2}}{K},$$

where we used $C^\ell/\ell! \ll C^n/n!$, valid for $0 \leq \ell \leq n \leq C$, and Stirling's formula. In the same way, if $K > e^{Bn}$ and B is large enough, then

$$\operatorname{meas}(E_n(K)) \ll \frac{n(4\log K)^n}{Kn!e^{Bn}} \ll \frac{(e^{-B/2}\log K)^n}{Kn!}.$$

Thus, we have

$$E(K) \leq \sum_{n=1}^{\infty} \operatorname{meas}(E_n(K)) \ll_\varepsilon K^{-1+e^{-B/2}}$$

and the Lemma follows. □

Corollary 8. *For $K \geq 1$, we have*

$$\operatorname{meas}(\{x \in [0,1] \mid |S(x)| > K\}) = O(e^{-\delta K}) \tag{3.4}$$

for some $\delta > 0$.

Proof. By Lemma 6 and 7, if $x \in [0,1] \setminus E(e^K)$ we have

$$S(x) \ll \sum_{n=1}^{\infty} \frac{B_\varepsilon n + K}{2^{n/2}} \ll_\varepsilon 1 + K$$

and (3.4) follows. □

We can now prove Theorem 2 and Corollary 1.

Proof of Theorem 2 and Corollary 1. Expressing the linear constraint in the definition of H_k as an integral, we see that

$$\begin{aligned} H_k &= (i\pi^2)^{-k} \lim_{X\to\infty} \int_0^1 \sum_{\substack{-X \leq n_1,\ldots,n_k \leq X,\\ n_1\cdots n_k \neq 0}} \frac{\mathrm{e}((n_1 + \cdots n_k)x) d(|n_1|) \cdots d(|n_k|)}{n_1 \cdots n_k}\, dx \\ &= \lim_{X\to\infty} \frac{2^k}{\pi^{2k}} \int_0^1 D_X(1,x)^k\, dx = \frac{2^k}{\pi^{2k}} \int_0^1 D_{\sin}(1,x)^k\, dx, \end{aligned} \tag{3.5}$$

where the exchange of order of summation and integration is justified by the dominated convergence theorem, since $D_X(1,x) \ll S(x)$ by Lemma 5 and, by (3.4),

$$\int_0^1 S(x)^k dx \le \sum_{L=1}^{\infty} \int_0^1 \chi_L(x) L^k dx \ll \sum_{L=1}^{\infty} L^k e^{-(L-1)\delta} \ll A^k k!,$$

for some $\delta, A > 0$ and where χ_L is the characteristic function of the set $\{x \mid L-1 \le S(x) \le L\}$. Since we also have $D_{\sin}(1,x) \ll S(x)$, the above computation also proves Theorem 2.

Finally, by (1.7) and (3.5) we have that for any fixed $k \in \mathbb{N}$ the k-th moment of $q^{-1}\,\mathrm{c}_0(a/q)$, for $a \in (\mathbb{Z}/q\mathbb{Z})^*$, $A_0 q < a < A_1 q$, and the k-th moment of $\frac{2}{\pi^2} D_{\sin}(1,x)$, for $x \in [0,1]$, are both equal to H_k. Since $\sum_{k=1} H_k t^k/k!$ has positive radius of convergence, then there exists a unique distribution function with moments H_k and so we obtain Corollary 2. □

Proof of Corollary 2. By Lemma 7, we can find some absolute constants K, B such that the set

$$S(x_1,x_2,\kappa) := \{x = \langle 0; 1, \dots, 1, x_1, x_2, a_{\kappa+3}, a_{\kappa+4}, \dots \rangle \mid 1 \le a_\ell \le K e^{B\ell}, \forall \ell \ge \kappa+3\}$$

has positive measure for any $x_1, x_2, \kappa \in \mathbb{Z}_{>0}$. Thus, to prove the corollary it is enough to show that for any $z \in \mathbb{R}$, $\varepsilon > 0$ there exist integers $x_1, x_2, \kappa \ge 1$ such that $z < D_{\sin}(1,x) \le z + \varepsilon$ for all $x \in S(x_1, x_2, \kappa)$.

Now, if $x \in S$ then by (1.8) we have $D_{\sin}(s,x) = \mathcal{A} + \mathcal{B} + \mathcal{C}$, where

$$\mathcal{A} = -\frac{\pi^2}{2} \sum_{\ell=1}^{\kappa} \frac{(-1)^\ell}{v_\ell} \bigg(\frac{1}{\pi v_\ell} + \psi\Big(\frac{v_{\ell-1}}{v_\ell}\Big)\bigg),$$

$$\mathcal{B} = -\frac{\pi^2}{2} \sum_{\ell=\kappa+1}^{\kappa+2} \frac{(-1)^\ell}{v_\ell} \bigg(\frac{1}{\pi v_\ell} + \psi\Big(\frac{v_{\ell-1}}{v_\ell}\Big)\bigg),$$

and, by (1.2) and Lemma 6,

$$\mathcal{C} \ll \sum_{\ell=\kappa+3}^{\infty} \frac{\log v_\ell}{v_{\ell-1}} \ll \sum_{\ell=\kappa+3}^{\infty} \frac{\log K + B\ell}{2^\ell} \le \varepsilon/10,$$

provided that $\kappa = \kappa_\varepsilon$ is large enough. Now, from the relation $v_n = a_n v_{n-1} + v_{n-2}$, we see that $v_{\kappa+1} = x_1 v_\kappa + v_{\kappa-1}$ and $v_{\kappa+2} = x_2(x_1 v_\kappa + v_{\kappa-1}) + v_\kappa$. Thus, by (1.2) we have

$$\begin{aligned} \mathcal{B} &= \alpha_\kappa \log(x_1) - \beta_\kappa \frac{\log(x_2)}{x_1 + \gamma_\kappa} + o(1) \\ &= \alpha_\kappa \log(x_1) - \beta_\kappa \frac{\log(x_2)}{x_1} + o(1) + O\bigg(\frac{\log(x_2)}{x_1^2}\bigg), \end{aligned}$$

as $x_1, x_2 \to \infty$ (and κ fixed), for some $\alpha_\kappa, \beta_\kappa, \gamma_\kappa \neq 0$. Now, if we pick

$$x_2 := [\exp\big(\beta_\kappa^{-1} x_1 (\alpha_\kappa \log(x_1) + \mathcal{A} - z - \varepsilon/2)\big)],$$

then $\mathcal{B} = z - \mathcal{A} + \varepsilon/2 + o(1)$. Thus, if x_1 is large enough we have

$$z < D_{\sin}(1, x) \le z + \varepsilon$$

and the corollary follows. □

Remark 5. *We remark that a modification of this proof in the spirit of* [Hic] *would have given Corollary 3 directly.*

Acknowledgments

After sending them a preprint of this paper, the author was informed by Maier and Rassias that they have also obtained independently Theorem 2 using a somewhat similar argument.

The author would like to thank Maksym Radziwiłł for useful discussions and Jens Marklof for suggesting Remark 3.

References

[Bag] Bagchi, Bhaskar. *On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann hypothesis.* Proc. Indian Acad. Sci. Math. 116 (2006), no. 2, 137-146; arxiv math.NT/0607733.

[BM] Balazard, M., Martin, B. *Sur l'autocorrélation multiplicative de la fonction ≪partie fractionnaire≫ et une fonction définie par J. R. Wilton.* Preprint, arxiv math.NT/1305.4395.

[BC] Bettin S.; Conrey, J.B. *A reciprocity formula for a cotangent sum.* Int. Math. Res. Not. IMRN, 2013, no. 24, 2. 5709–5726.

[dBT] de la Bretèche, R.; Tenenbaum, G. *Séries trigonomètriques à coefficients arithmétiques.* J. Anal. Math. 92, no. 1 (2004), 1–79.

[DM] Demirci Akarsu, E.; Marklof, J. *The value distribution of incomplete Gauss sums.* Mathematika 59 (2013), no. 2, 381–398.

[DFI] Duke, W.; Friedlander, J.B.; Iwaniec, H. *Bilinear forms with Kloosterman fractions.* Invent. Math. 128 (1997), no. 1, 23–43.

[Hic] Hickerson, D. *Continued fractions and density results for Dedekind sums.* J. Reine Angew. Math. 290 (1977), 113–116.

[Ish] Ishibashi, M. *The value of the Estermann zeta functions at* $s = 0$. Acta Arith. 73 (1995), no. 4, 357–361.

[Khi] Khinchin, A.Y. *Continued fractions.* The University of Chicago Press, Chicago, Ill.-London 1964.

[MR] Maier, H.; Rassias M.T. *Generalizations of a cotangent sum associated to the Estermann zeta function.* Preprint, arxiv math.NT/1410.2145

[Wil] Wilton, J.R. *An approximate functional equation with applications to a problem of Diophantine approximation.* J. Reine Angew. Math. 169 (1933), 219–237.

Sandro Bettin – Centre de Recherches Mathématiques - Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal, QC, H3C 3J7, Canada

E-mail address: bettin@crm.umontreal.ca