

# A NOTE ON THE DIMENSION OF THE LARGEST SIMPLE HECKE SUBMODULE

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ABSTRACT. For  $k \geq 2$  even, let  $d_{k,N}$  denote the dimension of the largest simple Hecke submodule of  $S_k(\Gamma_0(N); \mathbb{Q})^{\text{new}}$ . We show, using a simple analytic method, that  $d_{k,N} \gg_k \log \log N / \log(2p)$  with  $p$  the smallest prime co-prime to  $N$ . Previously, bounds of this quality were only known for  $N$  in certain subsets of the primes. We also establish similar (and sometimes stronger) results concerning  $S_k(\Gamma_0(N), \chi)$ , with  $k \geq 2$  an integer and  $\chi$  an arbitrary nebentypus.

## 1. INTRODUCTION

For an integral weight  $k \geq 2$  and a level  $N \geq 1$ , the anemic Hecke  $\mathbb{Q}$ -algebra

$$\mathbb{T} := \mathbb{Q}[T_n : (n, N) = 1],$$

generated by the Hecke operators  $T_n$ , acts on the space of cusp forms  $S_k(\Gamma_0(N))$ .

Simple Hecke submodules of  $S_k(\Gamma_0(N))$  of dimension  $d$  correspond to  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -orbits of size  $d$  of (arithmetically) normalized eigenforms  $f \in S_k(\Gamma_0(N))$ . When  $k = 2$ , the work of Shimura also gives a correspondence with simple factors of dimension  $d$  of the Jacobian  $J_0(N)$  of the modular curve  $X_0(N)$ . Thus it is interesting to ask about the dimension  $d_{k,N}$  of the largest simple Hecke submodule of  $S_k(\Gamma_0(N))$ , or equivalently the maximal degree of Hecke fields of normalized eigenforms.

Maeda [?] postulated that  $S_k(\Gamma_0(1))$  is a simple Hecke module for all even  $k \geq 12$ . This deep conjecture implies among other things that  $L(\frac{1}{2}, f) \neq 0$  for all  $f \in S_k(\Gamma_0(1))$ , see [?]. When  $N > 1$ , there is an obstruction to simplicity due to the Atkin–Lehner involutions, but numerical evidence suggests that this is the only asymptotic barrier when  $N$  is square-free. This led Tsaknias [?] to suggest the following generalization of Maeda’s conjecture (see also [?] for non-square-free levels):

**Conjecture 1.** *For  $k \geq 2$  even and large enough and  $N$  square-free, the number of Galois orbits of newforms in  $S_k(\Gamma_0(N))$  is  $2^{\omega(N)}$ . In particular, for any fixed  $\varepsilon > 0$  we have*

$$d_{k,N} \gg_{k,\varepsilon} N^{1-\varepsilon}.$$

*That is, there exists a constant  $c(k, \varepsilon) > 0$  depending at most on  $k$  and  $\varepsilon$  such that  $d_{k,N} > c(k, \varepsilon)N^{1-\varepsilon}$  for all square-free  $N \geq 1$ .*

There is a massive gap between Conjecture ?? and the unconditional results. Through an equidistribution theorem for Hecke eigenvalues, Serre [?] was the first to establish that  $d_{k,N} \rightarrow \infty$  as  $k + N \rightarrow \infty$ . Subsequently, by making Serre’s equidistribution theorem effective, Royer [?] and Murty–Sinha [?] showed that  $d_{k,N} \gg_{k,p} \sqrt{\log \log N}$  for any  $p \nmid N$ . In the particular case where  $N$  lies in a restricted set of primes, this bound has been improved by several authors. Extending a method of Mazur to all even weights, Billerey and Menares [?, Theorem 2] obtained that  $d_{k,N} \gg_k \log N$  when  $N \geq (k + 1)^4$  is in an explicit set of primes of lower natural density  $\geq 3/4$ . When the lower bound is fixed in advance and one looks for a level with a given number of prime divisors attaining it, see also [?]. When  $N \equiv 7 \pmod{8}$  is prime, Lipnowski–Schaeffer [?, Corollary 1.7] also showed that  $d_{2,N} \gg \log \log N$ , which can be significantly improved for  $N$  in certain subsets of the primes under certain well-known conjectures and heuristics.

In this paper we show that bounds of Lipnowski–Schaeffer quality can be obtained for all levels and integer weights. Our method is however, analytic and we believe simpler than the one in [?].

**Theorem 1.** *Let  $k \geq 2$  even and  $N \geq 1$  be integers. Then the dimension of the largest simple Hecke submodule of  $S_k(\Gamma_0(N))^{new}$  is*

$$d_{k,N} \gg_k \frac{\log \log N}{\log(2p_N)},$$

as  $N \rightarrow \infty$ , where  $p_N$  denotes the smallest prime co-prime to  $N$ .

Since the vast majority of integers  $N$  have a small co-prime factor, this bound is essentially asserting that  $d_{k,N} \gg_k \log \log N$ . Theorem ?? appears to be the first bound of “ $\log \log N$  strength” for any even weight  $k \geq 4$ , and in the case  $k = 2$ , without restriction on the level.

We state below a more general and precise form of Theorem ?? that holds in the presence of a nebentypus.

**Theorem 2.** *Let  $k \geq 2$  and  $N \geq 1$  be integers. Let  $p \nmid N$  and let  $\chi : (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}^\times$  be a homomorphism such that  $\chi(-1) = (-1)^k$ . Then the maximum size of the  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -orbits of newforms  $f \in S_k(\Gamma_0(N), \chi)$  is*

$$\geq \frac{2}{(k-1)\log(4p)} \cdot \log \left( \frac{\log N}{2\pi \log p} \right)$$

for all sufficiently large  $N$  (in terms of  $k$ ).

By definition, the same lower bound holds for the maximum degree of the Hecke fields  $K_f$  of newforms  $f$  (see Section ??). Note that  $K_f$  always contains the cyclotomic field  $\mathbb{Q}(\zeta_{\text{ord}(\chi)})$  generated by the values of  $\chi$  (a consequence of the Hecke relations at  $p^2$ , see Lemma ?? below), so the trivial lower bound in both cases is  $\varphi(\text{ord}(\chi))$ .

*Remark 1.* The result of Billerey–Menares mentioned above actually shows that when  $\ell \geq (k+1)^4$  belongs to an explicit set  $\mathcal{L}$  of primes with lower density  $\geq 3/4$ , there exists a normalized eigenform  $f \in S_k(\Gamma_0(\ell))^{\text{new}}$  with  $\deg(K_f) \gg_k \log \ell$ . Hence, for  $\varepsilon > 0$ , if an integer  $N$  has a prime factor  $\ell > N^\varepsilon$  that lies in  $\mathcal{L}$ , then  $\deg(K_f) \gg_{k,\varepsilon} \log N$  for some  $f \in S_k(\Gamma_0(\ell)) \hookrightarrow S_k(\Gamma_0(N))$ . Hence, Theorem ?? with “newform” replaced by the weaker conclusion “normalized eigenform” would follow from [?, Theorem 2] for almost all integers  $N$ .

In certain special situations it can be shown that the degree of the number field  $K_f$  is large for *all* newforms  $f \in S_k(\Gamma_0(N), \chi)$ . For instance, when  $p^r \mid N$  Brumer [?, p.3, Theorem 5.5, Remark 5.7] showed that  $K_f$  contains the maximal real subfield of the  $p^s$ -th roots of unity, where  $s = \lceil \frac{r}{2} - 1 - \frac{1}{p-1} \rceil$  (see also [?, ?]).

We exhibit a similar phenomenon which sometimes allows to significantly improve on Theorem ?? and the trivial bound  $\deg K_f \geq \varphi(\text{ord} \chi)$ , when  $k$  is odd, depending on the nebentypus  $\chi$  and the factorization of  $N$ .

**Theorem 3.** *Let  $k \geq 3$  be an odd integer,  $N \geq 1$  be square-free,  $\chi : (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}^\times$  be a homomorphism such that  $\chi(-1) = (-1)^k$ , and decompose*

$$N_2 = \prod_{\substack{p \mid N \\ \chi_p = 1}} p, \quad \chi = \prod_{p \mid N} \chi_p, \quad \text{with } \chi_p : (\mathbb{Z}/p)^\times \rightarrow \mathbb{C}^\times.$$

*Then, for any newform  $f \in S_k(\Gamma_0(N), \chi)$ ,*

$$\deg K_f \geq \varphi(\text{ord}(\chi)) \cdot 2^{\omega(N_2) - \omega((N_2, 2 \text{ord}(\chi))) - 1},$$

*In particular, if  $(N_2, 2 \text{ord}(\chi)) = 1$ , then*

$$\deg K_f \geq \varphi(\text{ord}(\chi)) \cdot 2^{\omega(N_2) - 1}.$$

For example, given  $\varepsilon > 0$  and  $k \geq 3$  odd, for a “typical” square-free integer  $N$  and  $\chi$  a random quadratic character mod  $N$  (resp. the trivial character), we get

$$\deg K_f \gg_\varepsilon (\log N)^{\frac{\log 2}{2} - \varepsilon} \quad (\text{resp. } \gg_\varepsilon (\log N)^{\log 2 - \varepsilon})$$

for all newforms  $f \in S_k(\Gamma_0(N), \chi)$ . In fact it is possible to extend Theorem ?? to the case of non-square-free  $N$ , but we maintain this restriction to keep the exposition simple.

*A short outline of the proofs.* We will now say a few words about the proof of these theorems and the limitations of our method of proof.

The proof of Theorem ?? and Theorem ?? proceeds by observing that if we can find a newform  $f$  for which the eigenvalue  $a_f(p_N)$  is abnormally small in absolute value but non-zero, then the degree of the corresponding Hecke field  $K_f$  needs to be large (see Proposition ??). We then use the equidistribution of Hecke eigenvalues (in the form of Murty–Sinha) to prove the existence of such an  $f$ . This contrasts with the previous analytic approaches in which one probed (using the equidistribution of Hecke eigenvalues) the neighborhood of every algebraic integer up to a certain height.

The proof of Theorem ?? proceeds by first noticing that by strong multiplicity one, the number field  $\mathbb{Q}(a_f(n) : n \geq 1)$  coincides with  $K_f = \mathbb{Q}(a_f(n) : (n, N) = 1)$ . Subsequently we focus exclusively on the ramified primes  $p \mid N$ . For  $k$  odd, the coefficient of  $f$  at  $p \mid N_2$  is equal to  $\sqrt{p}$  multiplied by a factor lying in a small extension of  $K_f$  (the eigenvalue of an Atkin–Lehner operator). Considering all these divisors yields the factor  $2^{\omega(N_2)}$ .

*Limitations of the method.* The best result that the method of proof of Theorem ?? and Theorem ?? can theoretically deliver is for each  $k$  even and  $N \geq 1$  the existence of an  $f \in S_k(\Gamma_0(N))$  such that  $\deg K_f \gg_k \log N$ . To see this consider for simplicity  $k$  fixed and  $N$  odd. Then we expect that the coefficients  $a_f(2)$  with  $f$  varying in  $S_k(\Gamma_0(N))$  behave as a collection of roughly  $\asymp_k N^{1+o(1)}$  random numbers distributed according to the Sato-Tate law. Therefore by linearity of expectation for any given  $\varepsilon > 0$  we expect that there exists a form  $f \in S_k(\Gamma_0(N))$  with  $0 < |a_f(2)| \ll_k N^{-1+\varepsilon}$  and moreover that this is best possible up to the factor  $N^\varepsilon$ . Plugging this into Proposition ?? would result in a lower bound  $\deg K_f \gg_k \log N$  for some  $f \in S_k(\Gamma_0(N))$ . Note that the existence of a  $\delta > 0$  such that for all  $k$  fixed and  $N$  odd there exists an  $f \in S_k(\Gamma_0(N))$  with  $0 < |a_f(2)| \ll_k N^{-\delta}$  would be also enough to obtain the lower bound  $\deg K_f \gg_k \log N$  for some  $f \in S_k(\Gamma_0(N))$ .

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## 2. PROOF OF THEOREM ?? AND THEOREM ??

Throughout let  $k \geq 2$  and  $N \geq 1$  be integers, and  $\chi : (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}^\times$  a homomorphism such that  $\chi(-1) = (-1)^k$ . Let  $f \in S_k(\Gamma_0(N), \chi)$  be a normalized eigenform

with Fourier expansion

$$f(z) := \sum_{n \geq 1} a_f(n) e(nz), \quad a_f(1) = 1, \quad e(z) := e^{2\pi iz}.$$

Given a prime  $p \nmid N$ , we also define (for reasons that will become clear when proving Lemma ??)

$$a'_f(p) = \frac{a_f(p)}{2p^{\frac{k-1}{2}} \sqrt{\chi(p)}} \in \mathbb{R},$$

for a fixed choice of square root.

Since simple Hecke submodules of  $S_k(\Gamma_0(N))$  of dimension  $d$  correspond to  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -orbits of size  $d$  of (arithmetically) normalized eigenforms  $f \in S_k(\Gamma_0(N))$  (see [?]), it suffices to obtain lower bounds for

$$\max_{f \in S_k(\Gamma_0(N), \chi)} \deg K_f, \quad K_f = \mathbb{Q}(a_f(n) : (n, N) = 1),$$

where  $f$  runs over newforms, to prove Theorems ?? and ??.

The first input to our argument is a simple lemma from diophantine approximation, that allows to pass from small values of  $|a_f(p)|$  to lower bounds for the degree of the Hecke field.

**Proposition 1.** *If  $p \nmid N$  and  $a_f(p) \neq 0$ , then*

$$\deg \mathbb{Q}(a_f(p)) \geq \frac{2}{k-1} \cdot \frac{\log \frac{1}{|a'_f(p)|}}{\log(4p)}.$$

*Proof.* Since  $a_f(p)$  is an algebraic integer [?, Corollary 12.4.5], its norm is a nonzero integer. Thus if we denote by  $g$  the degree of  $a_f(p)$  and by  $a_{f,1}(p), \dots, a_{f,g}(p)$  all of the conjugates of  $a_f(p)$  (including  $a_f(p)$  itself), then,

$$\prod_{i=1}^g |a_{f,i}(p)| \geq 1.$$

By Deligne's proof of the Ramanujan–Petersson conjecture for  $f$  [?],  $a_f(p)$  is the sum of two  $p$ -Weil numbers of weight  $k-1$ , so  $|a_{f,i}(p)| \leq 2p^{\frac{k-1}{2}}$  for all  $i$ . Therefore,

$$\prod_{i=1}^g |a_{f,i}(p)| \leq |a_f(p)| \left(2p^{\frac{k-1}{2}}\right)^{g-1}$$

and the claim follows. □

*Remark 2.* If  $\Gamma_f \leq \text{Gal}(K_f/\mathbb{Q})$  is the group of inner twists of  $f$  (see [?, Section 3], [?, Section 3]), then the proof of Proposition ?? shows that the lower bound can actually be improved by a factor of  $|\Gamma_f|$  (or even  $|\Gamma_f|^2$  if  $\chi(p) \in \mathbb{Q}^\times$ ). In the case

$k = 2$ ,  $\chi = 1$ ,  $N$  square-free, there are no nontrivial inner twists, but otherwise it is believed that  $|\Gamma_f|$  could become large; if  $\chi^2 \neq 1$ , there is always a nontrivial inner twist given by conjugation.

We will now use the equidistribution of Hecke eigenvalues to exhibit a newform  $f \in S_k(\Gamma_0(N), \chi)^{\text{new}}$  for which  $a_f(p)$  is abnormally small, yet non-zero. This will therefore give a lower bound for the degree of  $\mathbb{Q}(a_f(p))$  and thus a lower bound for the degree of  $K_f$ .

**Lemma 1.** *Let  $p \nmid N$ . There exists a newform  $f \in S_k(\Gamma_0(N), \chi)^{\text{new}}$  such that,*

$$(1) \quad 0 < |a'_f(p)| \leq \frac{\pi}{2} \cdot \frac{p+1}{p} \cdot \frac{\log p}{\log N}$$

for all sufficiently large  $N$  (in terms of  $k$ ).

*Proof.* Let  $B_k(\Gamma_0(N), \chi)$  be the  $\overline{\mathbb{Q}}$ -basis of  $S_k(\Gamma_0(N), \chi)^{\text{new}}$  composed of the  $d_{k,N,\chi}$  newforms at level  $N$ . For  $(n, N) = 1$ , let us also normalize Hecke operators acting on  $S_k(\Gamma_0(N), \chi)^{\text{new}}$  as  $T'_n := T_n / (2n^{\frac{k-1}{2}} \sqrt{\chi(n)})$ . By [?, Sections 5.1, 5.3], the normalized eigenvalues  $(a'_f(p))_{f \in B_k(\Gamma_0(N), \chi)}$  are distributed in  $[-1, 1]$  as  $N \rightarrow \infty$  according to a measure converging to the Sato–Tate measure as  $p \rightarrow \infty$ .

For  $A \in (0, 1)$ , let us give a lower bound on

$$C_{k,N,\chi}(A) := \frac{|\{f \in B_k(\Gamma_0(N), \chi) : 0 < |a'_f(p)| \leq A\}|}{d_{k,N,\chi}}.$$

If the nebentypus is trivial and we do not necessarily want to find a form that is new, we can directly apply [?, Theorem 19] to get (??) below. In general, [?, Theorem 8, Lemma 17, Section 10] show that for any  $M \geq 1$ ,

$$(2) \quad \left| C_{k,N,\chi}(A) - 2 \int_0^A F(-x) dx \right| \leq \frac{1}{M+1} + \sum_{1 \leq |m| \leq M} \left( \frac{1}{M+1} + \min \left( 2A, \frac{1}{\pi|m|} \right) \right) \left| \frac{\text{tr} \left( T'_{p^{|m|}} - T'_{p^{|m|-2}} \right)}{d_{k,N,\chi}} - c_m \right|,$$

where  $c_m = \lim_{k+N \rightarrow \infty} \text{tr}(T'_{p^{|m|}} - T'_{p^{|m|-2}}) / d_{k,N,\chi}$  and  $F(x) = \sum_{m \in \mathbb{Z}} c_m e(mx)$ , with the convention that  $T'_n = 0$  if  $n < 1$ . The Eichler–Selberg trace formula for  $S_k(\Gamma_0(N), \chi)$  [?, (34)] and [?, Section 5.3] gives that,

$$\begin{aligned} \text{tr } T'_{p^m} = & \sum_{N_1|N} d^*(N/N_1) \left( A_{\text{main}}(k, N_1, T'_{p^m}) + A_{\text{ell}}(k, N_1, \chi, T'_{p^m}) \right. \\ & \left. + A_{\text{hyp}}(k, N_1, \chi, T'_{p^m}) + \delta_{\substack{k=2 \\ \chi=1}} A_{\text{par}}(k, N_1, T'_{p^m}) \right), \end{aligned}$$

for any  $m \geq 1$ , with the main, elliptic, hyperbolic and parabolic terms given in [?, (35, 39, 45, 47)], and where  $d^*$  is the multiplicative function defined by  $d^*(\ell) = -2$ ,  $d^*(\ell^2) = 1$ , and  $d^*(\ell^\alpha) = 0$  for  $\ell$  a prime and  $\alpha \geq 3$  an integer. By [?, (35)],

$$\sum_{N_1|N} d^*(N/N_1) A_{\text{main}}(k, N_1, T'_{p^m}) = \frac{\psi(N)^{\text{new}}(k-1)}{12} \cdot p^{-m/2} \cdot \delta_{m \text{ even}},$$

where  $\psi(N)^{\text{new}} = \sum_{N_1|N} d^*(N/N_1) N_1 \prod_{\ell|N_1} (1 + 1/\ell)$ , and by [?, Section 9],

$$F(x) = \frac{\psi(N)^{\text{new}}(k-1)}{12d_{k,N,\chi}} \cdot \frac{2(p+1)}{\pi} \cdot \frac{\sqrt{1-x^2}}{p+2+1/p-4x^2}.$$

By [?, (44, 46, 48)], we find as in [?, (8)] that for any  $N_1 | N$ ,

$$\begin{aligned} |A_{\text{ell}}(k, N_1, \chi, T'_{p^m})| &\leq \frac{4e}{\log 2} \cdot 2^{\omega(N_1)} p^{3m/2} \log(4p^{m/2}), \\ |A_{\text{hyp}}(k, N_1, \chi, T'_{p^m}) - A_{\text{hyp}}(k, N_1, \chi, T'_{p^{m-2}})| &\leq \sqrt{N_1} \tau(N_1), \\ |A_{\text{par}}(k, N_1, T'_{p^m}) - A_{\text{par}}(k, N_1, T'_{p^{m-2}})| &\leq p^{m/2}. \end{aligned}$$

Moreover, we note that  $|d^*(n)| \leq 2^{\omega(n)} \leq \tau(n) \ll_\varepsilon n^\varepsilon$  for all integers  $n$  (see [?, (52)]). Hence, this yields with (??)

$$\begin{aligned} (3) \quad C_{k,N,\chi}(A) &\geq \frac{\psi(N)^{\text{new}}(k-1)}{12d_{k,N,\chi}} \cdot \frac{4(p+1)}{\pi} \int_0^A \frac{\sqrt{1-x^2}}{p+2+1/p-4x^2} dx - \frac{1}{M+1} \\ (4) \quad &- c(\varepsilon) N^\varepsilon \left( \frac{4e}{\log 2} \cdot \frac{p^{3M/2}}{d_{k,N,\chi}} \cdot \log(4p^{M/2}) - \frac{\sqrt{N}}{d_{k,N,\chi}} - \frac{\delta_{k=2}}{\chi=1} \cdot \frac{p^{M/2}}{d_{k,N,\chi}} \right), \end{aligned}$$

for any  $\varepsilon > 0$ , with  $c(\varepsilon) > 0$  a constant depending only on  $\varepsilon$ . As in [?, (61, 62)],

$$d_{k,N,\chi} = \frac{k-1}{12} \cdot \psi(N)^{\text{new}} + O(N^{1/2+\varepsilon}),$$

therefore given  $\varepsilon < 1/100$  positive, as long as  $M \leq (2/3 - 3\varepsilon) \log(N)/\log p$ , all the three terms in (??) are less than  $c'(\varepsilon) N^{-\varepsilon/100}$  for all  $N$  and some constants  $c'(\varepsilon)$  depending only on  $\varepsilon$ <sup>1</sup>.

By a Taylor expansion at  $x = 0$ ,

$$\int_0^A \frac{\sqrt{1-x^2}}{p+2+1/p-4x^2} dx = \frac{p}{(p+1)^2} \cdot A \cdot (1 + O(A)),$$

<sup>1</sup>The choice of  $M$  is motivated by the fact that the growth of (??) is dominated by the first term in (??) which is roughly of size  $N^\varepsilon p^{3M/2} N^{-1}$ . Thus it is sufficient to choose  $M$  so that this term is negligible, that is  $p^{3M/2} N^{\varepsilon-1} \ll N^{-\varepsilon}$ .

therefore

$$C_{k,N,\chi}(A) \geq \frac{4}{\pi} \cdot \frac{p}{p+1} \cdot A(1 + O(A)) - \frac{1}{M+1} - \frac{c'(\varepsilon)}{N^{\varepsilon/100}}.$$

Hence, given  $\varepsilon > 0$ , choosing  $A$  so that,

$$\frac{4}{\pi} \cdot \frac{p}{p+1} \cdot A > \left(\frac{3}{2} + \varepsilon\right) \cdot \frac{\log p}{\log N} > \frac{1 + \varepsilon}{M+1}$$

ensures that  $C_{k,N,\chi}(A) > 0$  for all sufficiently large  $N$ . In particular fixing a sufficiently small  $\varepsilon > 0$  we see that for all  $N$  large enough any

$$A > \frac{\pi}{2} \cdot \frac{p+1}{p} \cdot \frac{\log p}{\log N}$$

is acceptable. □

Theorem ?? and Theorem ?? now follows from combining Proposition ?? and Lemma ?? and specializing accordingly.

### 3. PROOF OF THEOREM ??

For  $k \geq 2$  and  $N \geq 1$  square-free, let  $f \in S_k(\Gamma_0(N), \chi)$  be a newform. We factor the character  $\chi$  as  $\prod_{p|N} \chi_p$  with  $\chi_p : (\mathbb{Z}/p)^\times \rightarrow \mathbb{C}^\times$  a character modulo  $p$ . The idea behind Theorem ?? is inspired by [?], where Choie and Kohnen show that the non-diagonalizability of a “bad” Hecke operator  $T_p$  (i.e. with  $p \mid N$ ) implies that  $\sqrt{p} \in \mathbb{Q}(a_n(f) : n \geq 1)$ , and hence that this field has degree at least  $2^s$  if  $s$  such operators are non-diagonalizable.

Let

$$N_2 = \prod_{\substack{p|N \\ \chi_p=1}} p$$

and write  $N = N_1 N_2$ , with  $(N_1, N_2) = 1$  since  $N$  is square-free. It follows that  $\chi = \chi_{N_1} \chi_{N_2}$  with  $\chi_{N_1}$  a primitive character of modulus  $N_1$  and  $\chi_{N_2} = 1$  the principal character modulo  $N_2$ . Our argument is based on the Atkin–Lehner operators

$$W_p : S_k(\Gamma_0(N), \chi) \rightarrow S_k(\Gamma_0(N), \bar{\chi}_p \chi_{N/p}), \quad p \mid N$$

where  $\chi_{N/p} = \prod_{\ell \mid N/p} \chi_\ell$  and on the properties of the pseudo-eigenvalues  $\lambda_p(f)$  studied by Atkin and Li [?, ?]. Examining these elements gives bounds on the degrees of Fourier coefficients  $a_f(p)$  at “bad” primes  $p \mid N_2$ . In turn, this yields lower bounds on  $\deg K_f$  since:

**Lemma 2.** *We have  $K_f = \mathbb{Q}(a_f(n) : n \geq 1)$ .*

*Proof.* Let  $K := \mathbb{Q}(a_f(n) : n \geq 1)$  and let  $L$  be its Galois closure. By the Hecke relations  $a_f(p)^2 = a_f(p^2) - p^{k-1}\chi(p)$  for all  $p \nmid N$ , we have the tower of extensions  $\mathbb{Q}(\zeta_{\text{ord } \chi}) \subset K_f \subset K \subset L$ . By Galois theory, it suffices to show that  $\text{Gal}(L/K_f) \subset \text{Gal}(L/K)$ . To that effect, let  $\sigma \in \text{Gal}(L/K_f)$ . By the fact that  $\chi^\sigma = \chi$  and [?, Corollary 12.4.5],  $f^\sigma$  is a newform in  $S_k(\Gamma_0(N), \chi)$  whose Fourier coefficients coincide with those of  $f$  at all integers co-prime to  $N$ . By strong multiplicity one [?, Theorem 6.2.3],  $f = f^\sigma$ , so that  $\sigma$  fixes all coefficients of  $f$ , i.e.  $\sigma$  fixes  $K$ .  $\square$

Recall that for  $p \mid N$ , the pseudo-eigenvalue  $\lambda_p(f) \in \mathbb{C}$  is defined by the equation

$$W_p f = \lambda_p(f)g,$$

where  $g \in S_k(\Gamma_0(N), \bar{\chi}_p \chi_{N/p})$  is a newform (see [?, p.224]) given by

$$(5) \quad a_g(\ell) = \begin{cases} \bar{\chi}_p(\ell)a_f(\ell) & : \ell \neq p \\ \chi_{N/p}(p)\overline{a_f(p)} & : \ell = p \end{cases}$$

for primes  $\ell$  ([?, (1.1)]).

In general, we only know that the pseudo-eigenvalue  $\lambda_p(f)$  is algebraic with modulus 1 ([?, Theorem 1.1]). However, under additional assumptions on  $\chi$ , we have the following information on its field of definition:

**Lemma 3.** *Let  $p \mid N_2$ . Then,  $\lambda_p(f) \in \mathbb{Q}(\zeta_{2 \text{ord}(\chi)})$ .*

*Proof.* From the identity  $W_p^2 = \chi_p(-1)\bar{\chi}_{N/p}(p)\text{id}$  ([?, Proposition 1.1]), we get that

$$(6) \quad \lambda_p(f)\lambda_p(g) = \chi_p(-1)\bar{\chi}_{N/p}(p) = \pm \bar{\chi}_{N/p}(p).$$

Since  $p \mid N_2$  we have  $\chi_p = 1$ , so that  $g \in S_k(\Gamma_0(N), \chi)$ , and  $a_g(\ell) = a_f(\ell)$  for all prime  $\ell \neq p$ , by (?). By strong multiplicity one, we get  $g = f$ . By (?), we obtain  $\lambda_p(f)^2 = \bar{\chi}_{N/p}(p)$  and thus the claim.  $\square$

The next ingredient is the explicit determination of  $\lambda_f(p)$  in terms of  $a_f(p)$  by Atkin and Li.

**Lemma 4.** *Let  $p \mid N_2$ . Then  $a_f(p) \neq 0$  and*

$$\lambda_p(f) = -\frac{p^{k/2-1}}{a_f(p)}.$$

*Proof.* The fact that  $a_f(p) \neq 0$  is [?, Theorem 3(ii)], and the formula for the eigenvalue is [?, Theorem 2.1].  $\square$

*Proof of Theorem ??.* By Lemmas ??, ?? and ??, we get

$$\{p^{k/2} : p \mid N_2\} \subset K_f(\zeta_{2 \text{ord}(\chi)}).$$

Since  $L := \mathbb{Q}(\zeta_{\text{ord}(\chi)}) \subset K_f$ , we have

$$\begin{aligned} [K_f : \mathbb{Q}] &\geq \frac{1}{2} \cdot [K_f(\zeta_{2\text{ord}(\chi)}) : \mathbb{Q}] \\ &= \frac{1}{2} \cdot [K_f(\zeta_{2\text{ord}(\chi)}) : L] \cdot \varphi(\text{ord}(\chi)), \end{aligned}$$

where the last factor is the trivial bound.

The square roots of odd primes  $p \mid \text{ord}(\chi)$  belong to  $L$ . On the other hand, for  $S := \{\sqrt{p} : p \mid N_2, p \nmid 2\text{ord}(\chi)\} \subset K_f(\zeta_{2\text{ord}(\chi)})$ , we have

$$[K_f(\zeta_{2\text{ord}(\chi)}) : L] \geq [L(S) : L] = 2^{|S|}$$

by [?, Theorem 87], and the claim follows.  $\square$

*Remark 3.* Since the character  $\chi_p$  is primitive for  $p \mid N_1$ , [?, Theorem 3(ii)] and [?, Theorem 2.1, Proposition 1.4] show that  $\lambda_p(f) = p^{k/2-1}g(\chi_p)/a_f(p)$ , with  $g(\chi_p)$  the Gauss sum attached to  $\chi_p$ . The degree of  $p^{k/2-1}g(\chi_p)$  over  $\mathbb{Q}$  can be determined precisely, however we have no information about the field of definition of  $\lambda_p(f)$ , except the fact that it is a root of unity. If we could show that it belongs to a small extension of  $K_f$ , in the same way as we did for  $\lambda_p(f)$  with  $p \mid N_2$ , then we could add a factor as large as  $\text{ord}(\chi)$  to the lower bound of Theorem ??, including when  $k$  is even.

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