Trilinear forms with Kloosterman fractions

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Abstract

We give new bounds for $\sum_{a,m,n} \alpha_m \beta_n \nu_a \, \mathrm{e}\big(\frac{a\overline{m}}{n}\big)$ where α_m, β_n and ν_a are arbitrary coefficients, improving upon a result of Duke, Friedlander and Iwaniec [DFI97]. We also apply these bounds to problems on representations by determinant equations and on the equidistribution of solutions to linear equations.

1 Introduction

For a, b, c positive integers, one defines the classical Kloosterman sum as

$$S(a, b; c) := \sideset{}{^*}\sum_{x \pmod c} \mathrm{e}\bigg(\frac{a\overline{x} + bx}{c}\bigg)$$

where, as usual, $\overline{x}$ denotes the multiplicative inverse of x modulo the denominator c, $\sum^*$ denotes a sum over the reduced residues modulo c, and $\mathrm{e}(x) := e^{2\pi i x}$.

Several important results in number theory have been obtained by using bounds for single Kloosterman sums such as Weil's bound, $S(a, b, c) \ll (a, b, c)^{\frac{1}{2}} c^{\frac{1}{2}+\varepsilon}$, or, more recently, for averages of Kloosterman sums, in particular the bounds of Deshouillers and Iwaniec [DI].

The results of [DI] are particularly efficient when considering averages of $S(a, b, c)$ with weights $f(a, b, c)$ that are smooth or have at least some special

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structure. For many applications, however, one would like to have non-trivial bounds in the case of arbitrary weights and such bounds would be useful also when the coefficients have arithmetic/geometric nature, but have conductor which is too large to be able to employ the extra information.

In the beautiful paper [DFI97], Duke, Friedlander and Iwaniec addressed this problem, obtaining a non-trivial bound for the following "bilinear form with Kloosterman fractions":

$$\mathcal{B}_a(M,N) := \sum\sum_{\substack{m\in\mathcal{M},n\in\mathcal{N},\\(m,n)=1}} \alpha_m\beta_n \,\mathrm{e}\bigg(\frac{a\overline{m}}{n}\bigg),$$

where α_m, β_n are arbitrary coefficients supported on $\mathcal{M} := [M/2, M]$ and $\mathcal{N} := [N/2, N]$ respectively, and $a \neq 0$. The main result of [DFI97] is the bound

$$\mathcal{B}_a(M,N) \ll \|\alpha\|\|\beta\|(|a| + MN)^{\frac{3}{8}}(M+N)^{\frac{11}{48}+\varepsilon}, \tag{1.1}$$

where $\|\cdot\|$ denotes the L^2-norm. Notice that, in the important case $M \approx N$, $a \ll MN$, the bound in (1.1) saves roughly a power of $N^{\frac{1}{48}}$ over the trivial bound $\mathcal{B}_a(M,N) \ll \|\alpha\|\|\beta\|(MN)^{\frac{1}{2}}$. In this paper, we refine the arguments of Duke, Friedlander and Iwaniec and improve upon their bound, obtaining

$$\mathcal{B}_a(M,N) \ll \|\alpha\|\|\beta\|(|a| + MN)^{\frac{1}{2}}(MN)^{-\frac{3}{20}+\varepsilon}(M+N)^{\frac{1}{4}}.$$

We get a saving of $N^{\frac{1}{20}}$ when $M \approx N$, $a \ll MN$. More generally, we consider the case when an extra average over a is introduced, as this often appears in applications. We then provide a new bound for sums of the form

$$\mathcal{B}(M,N,A) := \sum\sum\sum_{\substack{a\in\mathcal{A},m\in\mathcal{M},n\in\mathcal{N},\\(m,n)=1}} \alpha_m\beta_n\nu_a \,\mathrm{e}\bigg(\vartheta\frac{a\overline{m}}{n}\bigg),$$

where ν_a are arbitrary coefficients supported on $\mathcal{A} := [A/2, A]$ and $\vartheta \neq 0$.

Theorem 1. *Let ϑ be a nonzero integer. Then*

$$\begin{aligned}\mathcal{B}(M,N,A) \ll \|\alpha\|\|\beta\|\|\nu\|\Big(1 + \frac{|\vartheta|A}{MN}\Big)^{\frac{1}{2}}&\\ \times\Big((AMN)^{\frac{7}{20}+\varepsilon}(M+N)^{\frac{1}{4}} + (AMN)^{\frac{3}{8}+\varepsilon}(AN+AM)^{\frac{1}{8}}\Big).&\end{aligned} \tag{1.2}$$

Notice that when $|\vartheta|A \ll MN$ and $M \approx N$, the above bound improves the trivial bound $\mathcal{B}(M,N,A) \ll \|\alpha\|\|\beta\|\|\nu\|(AMN)^{\frac12}$ by roughly $\min(A^{\frac{3}{20}}N^{\frac{1}{20}}, N^{\frac18})$.

Remark 1. *One can perturb slightly the argument of the exponential function and still get the same bound. Indeed, if $f_{a,\vartheta}(x,y) \in \mathcal{C}^1(\mathbb{R}^2)$ is such that*

$$\frac{\partial}{\partial x} f_{a,\vartheta}(x,y) \ll \frac{X}{x^2 y}, \qquad \frac{\partial}{\partial y} f_{a,\vartheta}(x,y) \ll \frac{X}{xy^2}, \qquad \forall x \in \mathcal{N}, \forall y \in \mathcal{M} \quad (1.3)$$

for some $X > 1$ and any a, ϑ, then when $\vartheta \neq 0$ we have that

$$\mathcal{B}_f(M,N,A) := \sum_{m\in\mathcal{M}, n\in\mathcal{N}}\sum_{a\in\mathcal{A},}\sum_{(m,n)=1} \alpha_m\beta_n\nu_a \,\mathrm{e}\left(\vartheta\frac{a\overline{m}}{n} + f_{a,\vartheta}(m,n)\right)$$

satisfies the same bound (1.2) of $\mathcal{B}(M,N,A)$ provided that the factor $(1 + |\vartheta|A/(MN))^{\frac12}$ is replaced by $(1 + (|\vartheta|A + X)/(MN))^{\frac12}$.

Our main motivation for this paper concerns the second moment of the Riemann zeta-function times an arbitrary Dirichlet polynomial,

$$I := \int_{\mathbb{R}} \left|\zeta(\tfrac12 + it)A(\tfrac12 + it)\right|^2 \Phi(t/T)\,dt,$$

where $\Phi(x)$ is a test function (supported on $[1,2]$, say) and $A(s) := \sum_{n\le T^\theta} \frac{a_n}{n^s}$ with a_n arbitrary coefficients with $a_n \ll n^\varepsilon$. Balasubramanian, Conrey and Heath-Brown [BCH] computed the asymptotic for I when $\theta < \frac12$. In this case only the "diagonal terms" contribute. In order to break the $\frac12$ barrier one has to deal with the "off-diagonal terms", which quickly leads to the problem of obtaining non-trivial bounds for $\mathcal{B}(M,N,A)$. Equation (1.1) and the stronger Theorem 1 provide such non-trivial bounds, and so, in a joint work with Radziwiłł [BCR], we were able to compute the asymptotic for I for $\theta < \frac{17}{33}$. As a comparison, the use of (1.1) would have given the same result on the smaller range $\theta < \frac{48}{95}$. In the same work, we also formulate a conjectural bound for $\mathcal{B}$ which, if true, would allow to extend the range to $\theta < 1$ and thus imply the Lindelöf hypothesis (indeed, taking $a_n = n^{i\tau}$, $T = |\tau|$, and $N = T^{1-\varepsilon}$, one obtains $\zeta(\frac12 + i\tau) \ll (1 + |\tau|)^\varepsilon$ by using an easy modification of (7.20.2) in [Tit]).

The flexibility of Theorem 1 makes it feasible to be applied to a wide class of problems in number theory. Moreover, the strength of the new bound is now competitive even in some cases when one knows and could potentially employ some information on the coefficients.

We also give two easy applications of Theorem 1. One could use the new bound also to improve some results proved using (1.1) (e.g. [DFI12] or, possibly, to sharpen sub-convexity bounds for automorphic L-functions (see [DFI02]) or to handle sums such as those considered by Fouvry [Fou] and Bombieri, Friedlander and Iwaniec [BFI]. Another application was recently given in [BS], where our bound was used to study the distribution of the fractional parts of Dedekind sums.

The first corollary deals with representations by determinant equations and improves the main result of [DFI95], whereas the second concerns with the equidistribution of solutions to linear equations and improves upon a theorem of Shparlinski [Shp].

Corollary 1. *Let* $\Delta \neq 0$ *and let*

$$\mathcal{T}(M_1, M_2, N_1, N_2) := \sum\sum\sum\sum_{\substack{m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2, n_1 \in \mathcal{N}_1, n_2 \in \mathcal{N}_2, \\ m_1 n_2 - m_2 n_1 = \Delta}} f(m_1) g(m_2) \alpha_{n_1} \beta_{n_2},$$

where $f(m_1)$, $g(m_2)$, α_{n_1} *and* β_{n_2} *are supported on* $\mathcal{M}_1 := [M_1/2, M_1]$, $\mathcal{M}_2 := [M_2/2, M_2]$, $\mathcal{N}_1 := [N_1/2, N_1]$ *and* $\mathcal{N}_2 := [N_2/2, N_2]$ *respectively. Moreover, assume* $f^{(j)} \ll \eta^j M_1^{-j}, g^{(j)} \ll \eta^j M_2^{-j}$, *for all* $j \geq 0$ *and some* $\eta > 1$. *Then*

$$\mathcal{T}(M_1, M_2, N_1, N_2) = \sum_{\substack{n_1 \in \mathcal{N}_1, n_2 \in \mathcal{N}_2, \\ (n_1, n_2) | \Delta}} \frac{(n_1, n_2)}{n_1 n_2} \alpha_{n_1} \beta_{n_2} \int_{\mathbb{R}} f\left(\frac{x + \Delta}{n_2}\right) g\left(\frac{x}{n_1}\right) dx$$
$$+ O\left((\eta R)^{\frac{3}{2}} \|\alpha\| \|\beta\| (N_1 N_2)^{\frac{7}{20}} (N_1 + N_2)^{\frac{1}{4} + \varepsilon} (M_1 M_2)^{\varepsilon} \right), \tag{1.4}$$

where $R := \frac{M_1 N_2}{M_2 N_1} + \frac{M_2 N_1}{M_1 N_2}$.

For comparison, Duke, Friedlander and Iwaniec [DFI95] obtained the same result with the error term

$$O\left((\eta R)^{\frac{19}{8}} \|\alpha\| \|\beta\| (N_1 N_2)^{\frac{3}{8}} (N_1 + N_2)^{\frac{11}{48} + \varepsilon} (M_1 M_2)^{\varepsilon} \right).$$

We remark that one can use Theorem 1 to obtain stronger results when averaging over Δ (cf. [BCR]).

Corollary 2. *For any positive coprime integers m, n, let $\rho_{m,n} := \frac{a_0}{m}$ where (a_0, b_0) is the smallest positive solution to $am - bn = 1$. For any set of integers $\mathcal{X}_N \subseteq [0, N]$, let $\mathcal{R}_{\mathcal{X}_N} := \{\rho_{m,n} \mid m, n \in \mathcal{X}_N\}$. Then, if $\mathcal{X}_N$ has cardinality $|\mathcal{X}_N| \gg N^{1-\frac{1}{20}}$ for some $\varepsilon > 0$, then the set $\mathcal{R}_{\mathcal{X}_N}$ is equidistributed on the interval $[0, 1]$ as $N \to \infty$.*

Shparlinski obtained the same result with $\frac{1}{20}$ replaced by $\frac{1}{48}$. We skip the proof of this corollary as it can be obtained in a straightforward manner by proceeding as in [Shp], using (1.2) instead of (1.1).

As observed in [DFI97], a variation of the arguments used to bound $\mathcal{B}(M, N, A)$ can be used to treat the twisted sum

$$\mathscr{A}(M, N, A) := \sum_{\substack{a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, \\ (m,n)=(2,mn)=1}} \sum \sum \alpha_m \beta_n \nu_a \Big(\frac{m}{n}\Big) \, \mathrm{e}\Big(\vartheta \frac{a\overline{m}}{n}\Big),$$

where $\big(\frac{\cdot}{\cdot}\big)$ is the Jacobi symbol. We thus conclude the introduction with the analogue of Theorem 1 for $\mathscr{A}(M, N, A)$.

Theorem 2. *Let $\vartheta \neq 0$. Then*

$$\begin{aligned} \mathscr{A}(M, N, A) \ll \|\alpha\| \|\beta\| \|\nu\| \Big(1 + \frac{|\vartheta| A}{NM}\Big)^{\frac{1}{2}} \\ \times \Big((MN)^{\frac{3}{10}} (AM + AN)^{\frac{7}{20} + \varepsilon} + A^{\frac{1}{2}} (N + M)^{\frac{7}{8} + \varepsilon}\Big). \end{aligned} \tag{1.5}$$

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2 Outline of the proof of Theorem 1

Our proof has roughly the same structure of Duke, Friedlander and Iwaniec's proof of (1.1) and follows their clever application of the amplification method. However we introduce several refinements in their arguments, among which is particularly important the fact that we keep a longer diagonal when using the Cauchy-Schwartz inequality (a possibility mentioned in [DFI97]). This change, together with the extra average over a, introduces new subtle complications and requires a rather careful analysis.

We now give an outline of the proof of Theorem 1. The proof of Theorem 2 is very similar and the required changes are described in Section 8.

First, we notice that we can assume that β_n is supported on integers coprime to ϑ, as can be seen by pulling out the common factor between n and ϑ and applying the Cauchy-Schwarz inequality. Then, following [DFI97], we apply the Cauchy-Schwarz inequality to the sum over m and obtain that

$$\mathcal{B}(M,N,A) \ll \|\alpha\| \mathcal{C}_1(M,N,A;\beta,\nu)^{\frac{1}{2}}, \tag{2.1}$$

where for any positive integer b,

$$\mathcal{C}_b(M,N,A;\beta,\nu) := \sum_{\substack{m\in\mathcal{M},\\(m,b)=1}} \Bigg| \sum_{a\in\mathcal{A}} \sum_{\substack{n\in\mathcal{N},\\(m,n)=1}} \beta_n \nu_a \,\mathrm{e}\Big(\vartheta\frac{a\overline{m}}{bn}\Big)\Bigg|^2.$$

At this point, we assume that β_n is supported on square-free integers coprime to b and that $(\vartheta,b)=1$. We will first give a bound for $\mathcal{C}_b$ in this case and in Section 6 we will use the freedom given by the parameter b to obtain a bound for $\mathcal{C}_1$ valid in the general case.

We introduce an amplifier and consider the sum $\mathcal{D}_b(M,N,A,L;\beta,\nu)$ defined by

$$\mathcal{D}_b := \sum_{\substack{m\in\mathcal{M},\\(m,b)=1}} \frac{1}{\varphi(m)} \sum_{\chi \pmod m} \Big| \sum_{\substack{\ell\in\mathcal{L},\\(\ell,\vartheta b)=1}} \chi(\ell)\Big|^2 \Bigg| \sum_{\substack{n\in\mathcal{N},a\in\mathcal{A},\\(m,n)=1}}\sum \chi(n)\beta_n\nu_a \,\mathrm{e}\Big(\vartheta\frac{a\overline{m}}{bn}\Big)\Bigg|^2,$$

where $\mathcal{L} := \{\ell \text{ is prime} \mid L<\ell<2L\}$ and L is a parameter to be chosen at the end of the argument. If $L > 2\log(b\vartheta M)$, then

$$\sum_{\ell\in\mathcal{L},\ (\ell,\vartheta b)=1} \chi_0(\ell) \gg \frac{L}{\log L},$$

where χ_0 is the principal character modulo m. Thus,

$$\begin{aligned}\mathcal{C}_b &\ll \frac{M\log^2 L}{L^2} \sum_{\substack{m\in\mathcal{M},\\(m,b)=1}} \frac{1}{\varphi(m)} \Bigg| \sum_{\substack{(\ell,\vartheta b)=1,\\ \ell\in\mathcal{L}}} \chi_0(\ell)\Bigg|^2 \Bigg| \sum_{\substack{n\in\mathcal{N},\\(m,n)=1}} \sum_{a\in\mathcal{A}} \beta_n\nu_a \,\mathrm{e}\Big(\vartheta\frac{a\overline{m}}{bn}\Big)\Bigg|^2 \\ &\ll ML^{-2+\varepsilon}\mathcal{D}_b(M,N,A,L;\beta,\nu),\end{aligned} \tag{2.2}$$

provided that $L > 2\log(b\vartheta M)$. Thus, we have reduced the problem of bounding $\mathcal{C}_b$ to that of bounding the more flexible $\mathcal{D}_b$. Squaring out and exploiting the orthogonality relation of character sums, we obtain

$$\begin{aligned}\mathcal{D}_b(M,N,A;\beta,\nu) &= \sum\sum\sum\sum\sum\sum\sum_{\substack{m\in\mathcal{M},n_1,n_2\in\mathcal{N},a_1,a_2\in\mathcal{A},\ell_1,\ell_2\in\mathcal{L},\\ (mb\vartheta,\ell_1\ell_2n_1n_2)=(m,b)=1,\\ \ell_1n_1\equiv\ell_2n_2 \pmod m}} \beta_{n_1}\nu_{a_1}\overline{\beta_{n_2}\nu_{a_2}}\,\mathrm{e}\left(\vartheta\frac{a_1\overline{m}}{bn_1}-\vartheta\frac{a_2\overline{m}}{bn_2}\right)\\ &= \mathscr{D}_b(M,N,A,L;\beta,\nu)+\mathscr{O}_b(M,N,A,L;\beta,\nu), \qquad (2.3)\end{aligned}$$

where $\mathscr{D}_b$ is the contribution to $\mathcal{D}_b$ from the "diagonal terms" $\ell_1n_1=\ell_2n_2$, and $\mathscr{O}_b$ is the sum restricted to the "off-diagonal" terms $\ell_1n_1\neq\ell_2n_2$.

We bound $\mathscr{D}_b$ in Section 3 by using Weil's bound (and thus the name "diagonal terms" is perhaps misleading in this case), treating it differently from [DFI97] where $\mathscr{D}_b$ is bounded trivially.

Section 4 is devoted to bounding $\mathscr{O}_b$ which constitutes the most delicate part of the proof. The argument proceeds roughly in the following way (where the bounds stated below refer to the case when ϑ, A and b are not too big with respect to MN)

- In Section 4.1.1, we switch to the complementary divisor d of the congruence relation $\ell_1n_1\equiv\ell_2n_2 \pmod m$, eliminating the variable m. This also requires that we first split the sum over m into congruence classes $c \pmod b$. Notice that the divisor switching makes the conductor drop from size M to size $D=NL/M\approx L$ when $N\approx M$.

- In Section 4.1.2, we apply the Cauchy-Schwarz inequality to the sums over n_1,n_2,a_2,c but not to the sums over d,a_1,ℓ_1,ℓ_2. As a comparison, in [DFI97] the Cauchy-Schwarz inequality is applied to all the sums except those over ℓ_1 and ℓ_2. After this operation we obtain that $\mathscr{O}_b$ is bounded by $b^{\frac{1}{2}}\|\beta\|^2\|\nu\|$ times the square-root of a sum which has roughly the following shape:

$$\sum_{\substack{n_1,n_2\asymp N,\\ \ell_1,\ell_2,\ell_1',\ell_2'\asymp L,\\ 0\neq|d_1|,|d_1|\ll D,\\ a_1,a_1',a_2\asymp A}} \nu_{a_1}\overline{\nu}_{a_1'} \sideset{}{^*}\sum_{\substack{c \pmod b,\\ \ell_1n_1\equiv\ell_2n_2-cd \pmod{b|d|},\\ \ell_1'n_1\equiv\ell_2'n_2-cd' \pmod{b|d'|}}} \mathrm{e}\left(\delta_1\frac{\overline{\ell_2\ell_2'bn_2}}{n_1}+\delta_2\frac{\overline{\ell_1\ell_1'bn_1}}{n_2}\right),$$

  where $\delta_1=-\vartheta(da_1\ell_2'-d'a_1'\ell_2)$, $\delta_2=-\vartheta a_2(d\ell_1'-d'\ell_1)$ and $D=NL/M\approx L$ if $N\approx M$. (Notice that we lost a factor of $b^{\frac{1}{2}}$ over the trivial bound when applying Cauchy-Schwarz with respect to c.)

- In Section 4.1.3.2, we apply the elementary reciprocity law (4.18), which roughly allows one to change $\frac{\overline{b\alpha}}{\beta}$ into $-\frac{\overline{b\beta}}{\alpha} - \frac{\overline{\alpha\beta}}{b}$ modulo 1, and so the above exponential becomes $\mathrm{e}\Big(\delta_1 \frac{\overline{\ell_2\ell_2' bn_2}}{n_1} + \delta_2 \frac{\overline{bn_2}}{\ell_1\ell_1' n_1} + \frac{\phi}{b}\Big) = \mathrm{e}\Big(\Delta \frac{\overline{\ell_2\ell_2' bn_2}}{\ell_1\ell_1' n_1} + \frac{\phi}{b}\Big)$, where $\Delta = a_2(d\ell_1' - d'\ell_1)\ell_2\ell_2' - (da_1\ell_2' - d'a_1'\ell_2)\ell_1\ell_1'$ and ϕ doesn't depend on n_2 (since the above congruence conditions determine n_2 modulo b).

- If $\Delta \neq 0$, then we use the congruence conditions to determine $c \pmod b$ (independently from n_2) and we apply Weil's bound to the sum over n_2. This way we gain a factor of $N^{\frac{1}{2}}/L$ and lose a factor of D^2 (since the congruence conditions modulo d, d' get lost) over the trivial bound for the above sum. Thus, the contribution of the $\Delta \neq 0$ terms to $\mathscr{O}_b$ is $\ll b^{\frac{1}{2}}\|\beta\|^2\|\nu\|^2 L^{\frac{7}{2}} N^{\frac{3}{4}} A$ if $N \approx M$.

- If $\Delta = 0$, then from this equality we can express two out of the ℓ and one out of the a variables in terms of the remaining variables, saving (the square root of) a factor of $L^2 A$ so that the contribution of the $\Delta = 0$ terms to $\mathscr{O}_b$ is $\ll b^{\frac{1}{2}}\|\beta\|^2\|\nu\|^2 LNA^{\frac{1}{2}}$ if $N \approx M$. We remark that here we are able to use fully the information given by the condition $\Delta = 0$. Indeed, interpreting it as a congruence modulo ℓ_i, ℓ_i' we can determine two of this variables (as in [DFI97]), whereas interpreting it as an equation we can determine one among a_1, a_1';[1] it is also important that we can do this without losing the congruence conditions modulo d_1, d_2.

In Section 5 we combine the bounds obtained for the diagonal and off-diagonal terms, and we optimize the parameter L. If $N \approx M$ and ϑ, A and b are not too big, then the main contribution come from the off-diagonal terms, so from (2.2) we obtain $\mathcal{C}_b \ll b^{\frac{1}{2}}\|\beta\|^2\|\nu\|^2 N^{\frac{7}{4}+\varepsilon} A^{\frac{1}{2}}(A^{\frac{1}{2}}L^{\frac{3}{2}} + L^{-1}N^{\frac{1}{4}})$. Equalizing $A^{\frac{1}{2}}L^{\frac{3}{2}}$ and $L^{-1}N^{\frac{1}{4}}$ we are led to take $L \approx A^{-\frac{1}{5}}N^{\frac{1}{10}}$ and so $\mathcal{C}_b \ll b^{\frac{1}{2}}\|\beta\|^2\|\nu\|^2 N^{\frac{19}{10}+\varepsilon} A^{\frac{7}{10}}$. Inserting this in (2.1) we obtain Theorem 1 in the case where β_n is supported on square-free integers, $N \approx M$ and ϑ, A and b not too big.

Finally, in Section 6 we remove the square-free condition on β_n, and in Section 7 we complete the proof of Theorem 1.

When following the above steps, several complications arise when some of the variables are not pairwise coprime or are far from the "main" ranges.

[1] In particular not applying Cauchy-Schwarz with respect to a_2 would not help, since we can use $\Delta = 0$ as an equation to determine a variable only once.

Typically, we first deal with the case when the variables are coprime and then we remove these assumptions by using the bounds proved in the coprime case.

As mentioned above, we conclude the paper by explaining, in Section 8, the modifications needed in the above arguments to prove Theorem 2 and, in Section 9, we give a proof to Corollary 1.

Remark. *Throughout the paper, we use the common convention in analytic number theory that ε denotes an arbitrarily small positive quantity that may vary from line to line.*

3 The diagonal terms

In this section we bound the diagonal terms $\mathscr{D}_b := \mathscr{D}_b(M, N, A, L; \beta, \nu)$ as the following lemma.

Lemma 1. *Assume that $(\vartheta, b) = 1$ and that β_n is supported on integers which are coprime to b. Also let $A, b, \vartheta, N \ll M^C$ for some constant $C > 0$. Then*

$$\mathscr{D}_b(M, N, A, L; \beta, \nu) \ll \|\beta\|^2 \|\nu\|^2 L\Big(A(bLN)^{\frac{1}{2}} + \frac{AM}{bN} + M\Big)M^{\varepsilon}. \tag{3.1}$$

Proof. By symmetry and the inequality $2|ab| \le a^2 + b^2$, we see that the diagonal terms are bounded by

$$\begin{aligned}\mathscr{D}_b :=& \sum_{\substack{m\in\mathcal{M},\ell_1,\ell_2\in\mathcal{L},n_1,n_2\in\mathcal{N},a_1,a_2\in\mathcal{A}\\ (mb\vartheta,\ell_1\ell_2n_1n_2)=(m,b)=1,\ \ell_1n_1=\ell_2n_2}}\sum\sum\sum\sum\sum\sum \nu_{a_1}\overline{\nu_{a_2}}\beta_{n_1}\overline{\beta}_{n_2}\, \mathrm{e}\Big(\vartheta\frac{a_1\overline{m}}{bn_1} - \vartheta\frac{a_2\overline{m}}{bn_2}\Big)\\ &\le \sum_{\substack{\ell_1,\ell_2\in\mathcal{L},n_1,n_2\in\mathcal{N},a_1,a_2\in\mathcal{A}\\ (b\vartheta,\ell_1\ell_2n_1n_2)=1,\ \ell_1n_1=\ell_2n_2}}\sum\sum\sum\sum\sum (|\beta_{n_1}\nu_{a_1}|^2 + |\beta_{n_2}\nu_{a_2}|^2)\\ &\qquad\qquad \times \Bigg|\sum_{\substack{m\in\mathcal{M},\\ (m,b\ell_1\ell_2n_1n_2)=1}} \mathrm{e}\Big(\vartheta\frac{a_1\ell_1\overline{m}}{b\ell_1n_1} - \vartheta\frac{a_2\ell_2\overline{m}}{b\ell_2n_2}\Big)\Bigg|\\ &\ll \sum_{\substack{\ell_1,\ell_2\in\mathcal{L},n_1,n_2\in\mathcal{N},a_1,a_2\in\mathcal{A}\\ (b\vartheta,\ell_1\ell_2n_1n_2)=1,\ \ell_1n_1=\ell_2n_2}}\sum\sum\sum\sum\sum |\beta_{n_1}\nu_{a_1}|^2 \Bigg|\sum_{\substack{m\in\mathcal{M},\\ (m,b\ell_1\ell_2n_1n_2)=1}} \mathrm{e}\Big(\vartheta\frac{(a_1\ell_1 - a_2\ell_2)\overline{m}}{b\ell_1n_1}\Big)\Bigg|.\end{aligned} \tag{3.2}$$

For the terms satisfying $a_1\ell_1 \neq a_2\ell_2$ we use the version of Weil's bound given in Lemma 3, in the appendix. We obtain

$$\sum_{\substack{m\in\mathcal{M},\\(m,b\ell_1\ell_2n_1n_2)=1}} \mathrm{e}\bigg(\vartheta\frac{(a_1\ell_1-a_2\ell_2)\overline{m}}{b\ell_1 n_1}\bigg) \ll (bLN)^{\frac{1}{2}+\varepsilon} + (a_1\ell_1-a_2\ell_2, bn_1\ell_1)\frac{M^{1+\varepsilon}}{bLN},$$

since $(\vartheta, b\ell_1 n_2)=1$. It follows that the contribution to $\mathscr{D}_b$ coming from these terms is bounded by

$$\begin{aligned}
&\ll \sum_{\ell_1,\ell_2\in\mathcal{L},}\sum_{n_1\in\mathcal{N},}\sum_{a_1,a_2\in\mathcal{A},}\sum\sum_{\ell_2|\ell_1n_1} |\beta_{n_1}\nu_{a_1}|^2(bLN)^{\frac{1}{2}+\varepsilon} + \\
&\quad + \sum_{n_1\in\mathcal{N},}\sum\sum_{a_1,a_2\in\mathcal{A}} |\beta_{n_1}\nu_{a_1}|^2\frac{M^{1+\varepsilon}}{bLN} \sum\sum_{\substack{\ell_1,\ell_2\\ a_1\ell_1\neq a_2\ell_2,\ \ell_2|\ell_1n_1}} (a_1\ell_1-a_2\ell_2, bn_1)(a_2\ell_2,\ell_1)\\
&\ll \|\beta\|^2\|\nu\|^2 ALM^{\varepsilon}\Big((bLN)^{\frac{1}{2}}+\frac{M}{bN}\Big),
\end{aligned}$$

since

$$\begin{aligned}
\sum\sum\sum_{\substack{\ell_1,\ell_2,a_2,\\ a_1\ell_1\neq a_2\ell_2,\ \ell_2|\ell_1n_1}} (a_1\ell_1-a_2\ell_2, bn_1) &\ll \sum_{\substack{|c|\le 4AL,\\ c\neq 0}} (c,bn_1) \sum\sum\sum_{\substack{\ell_1,\ell_2,a_2,\\ a_1\ell_1-a_2\ell_2=c,\\ \ell_2|\ell_1n_1}} (a_2\ell_2,\ell_1) \qquad (3.3)\\
&\ll AL^2M^{\varepsilon}.
\end{aligned}$$

The contribution to $\mathscr{D}_b$ coming from the terms with $a_1\ell_1 = a_2\ell_2$ is trivially $O(\|\beta\|^2\|\nu\|^2LM^{1+\varepsilon})$, and we thus obtain (3.1).

□

4 The off-diagonal terms

In this section we bound the off-diagonal terms $\mathscr{O}_b(M,N,A,L;\beta,\nu)$. We start by dividing $\mathscr{O}_b$ according to whether $(\ell_1,\ell_2)=1$ or not:

$$\mathscr{O}_b(M,N,A,L;\beta,\nu) = \mathscr{E}_{b,1}(M,N,A,L;\beta,\nu) + \mathscr{E}^*_{b,1}(M,N,A,L;\beta,\nu), \quad (4.1)$$

where, for any η satisfying $(\eta,b)=1$ and $(\eta,\ell)=1$ for all $\ell\in\mathcal{L}$, we define

$$\mathscr{E}_{b,\eta} := \sum_{\substack{\ell_1,\ell_2\in\mathcal{L},m\in\mathcal{M},n_1,n_2\in\mathcal{N},a_1,a_2\in\mathcal{A}\\ (mb\vartheta,\ell_1\ell_2n_1n_2)=(m,b)=(\ell_1,\ell_2)=1,\ \eta|m,\\ \ell_1n_1\equiv\ell_2n_2 \pmod{m},\ \ell_1n_1\neq\ell_2n_2}} \sum\sum\sum\sum\sum\sum \beta_{n_1}\nu_{a_1}\overline{\beta_{n_2}\nu_{a_2}}\,\mathrm{e}\bigg(\vartheta\frac{a_1\overline{m}}{bn_1}-\vartheta\frac{a_2\overline{m}}{bn_2}\bigg), \quad (4.2)$$

and $\mathscr{E}^*_{b,\eta}(M,N,A,L;\beta,\nu)$ is the same sum with $(\ell_1,\ell_2)=1$ replaced by $(\ell_1,\ell_2)>1$. We notice that

$$\mathscr{E}^*_{b,1} = \sum_{1<\ell\in\mathcal{L}} (\mathscr{E}_{b,1}(M,N,A,\{1\};\beta,\nu) - \mathscr{E}_{b,\ell}(M,N,A,\{1\};\beta,\nu)), \tag{4.3}$$

where we extended the definition of $\mathscr{E}_{b,\eta}$ to the case where $\mathcal{L}=\{1\}$. Thus it suffices to bound $\mathscr{E}_{b,\eta}$ as in the following Lemma.

Lemma 2. *Assume that $(\vartheta,b)=1$ and that β_n is supported on square-free integers which are coprime to b. Also let $A,b,\vartheta,N \ll M^C$ for some constant $C>0$. Then*

$$\mathscr{E}_{b,\eta} \ll \|\beta\|^2\|\nu\|^2\Big(b+\frac{|\vartheta|A}{NM}\Big)^{\frac12} ALN^{\frac34}M^{\varepsilon}\bigg(\frac{b^{\frac14}N^{\frac12}L^{\frac12}}{M^{\frac12}}+\frac{L^{\frac52}N}{M}+\frac{N^{\frac14}}{A^{\frac12}}\bigg).$$

It then follows that

$$\mathscr{E}^*_{b,1} \ll \|\beta\|^2\|\nu\|^2\Big(b+\frac{|\vartheta|A}{NM}\Big)^{\frac12} ALN^{\frac34}M^{\varepsilon}\bigg(\frac{b^{\frac14}N^{\frac12}}{M^{\frac12}}+\frac{N}{M}+\frac{N^{\frac14}}{A^{\frac12}}\bigg),$$

and so, by (4.1), we obtain the following bound for $\mathscr{O}_b$.

Corollary 3. *Under the same conditions of Lemma 2 we have*

$$\mathscr{O}_b \ll \|\beta\|^2\|\nu\|^2\Big(b+\frac{|\vartheta|A}{NM}\Big)^{\frac12} ALN^{\frac34}M^{\varepsilon}\bigg(\frac{b^{\frac14}N^{\frac12}L^{\frac12}}{M^{\frac12}}+\frac{L^{\frac52}N}{M}+\frac{N^{\frac14}}{A^{\frac12}}\bigg). \tag{4.4}$$

The remaing part of Section 4 is devoted to proving Lemma 2.

We start by introducing some notation:

$$\begin{aligned} n_1' &:= \frac{n_1}{(n_1,\ell_1\ell_2)} = \frac{n_1}{\mathfrak{p}_1\mathfrak{p}_2}, \quad \mathfrak{p}_2=(\ell_2,n_1), \quad \mathfrak{p}_1=(\ell_1,n_1),\\ n_2' &:= \frac{n_1}{(n_2,\ell_1\ell_2)} = \frac{n_2}{\mathfrak{q}_1\mathfrak{q}_2}, \quad \mathfrak{q}_1=(\ell_1,n_2), \quad \mathfrak{q}_2=(\ell_2,n_2), \end{aligned} \tag{4.5}$$

and notice that, for square-free integers n_1,n_2, this automatically gives $(\ell_1\ell_2,n_1'n_2')=1$. We also divide $\mathscr{E}_{b,\eta}(M,N,A,L;\beta,\nu)$ further into

$$\mathscr{E}_{b,\eta} = \mathscr{S}_{b,\eta} + \mathscr{S}^*_{b,\eta}, \tag{4.6}$$

where $\mathscr{S}_{b,\eta}$ and $\mathscr{S}^*_{b,\eta}$ are obtained by restricting the sums to $(n_1',n_2')=1$ and $(n_1',n_2')>1$ respectively. We will prove a bound for $\mathscr{S}_{b,\eta}$ in Subsection 4.1 (cf. equation (4.32)), and in Subsection 4.2 we will deduce from this bound a bound for $\mathscr{S}^*_{b,\eta}$.

4.1 The terms with $(n_1', n_2') = 1$

In this section we consider the sum

$$\mathscr{S}_{b,\eta} := \sum_{\substack{\ell_1,\ell_2\in\mathcal{L}, n_1,n_2\in\mathcal{N}, m\in\mathcal{M}, a_1,a_2\in\mathcal{A} \\ (mb\vartheta,\ell_1\ell_2n_1n_2)=(m,b)=(\ell_1,\ell_2)=(n_1',n_2')=1 \\ \ell_1n_1\equiv\ell_2n_2 \pmod m,\ (\ell_1\ell_2,n_1'n_2')=1, \\ \ell_1n_1\neq\ell_2n_2,\ \eta|m}} \sum\sum\sum\sum\sum\sum\sum \beta_{n_1}\nu_{a_1}\overline{\beta_{n_2}\nu_{a_2}}\,\mathrm{e}\left(\vartheta\frac{a_1\overline{m}}{bn_1} - \vartheta\frac{a_2\overline{m}}{bn_2}\right).$$

4.1.1 Introducing the complementary divisor

We wish to switch to the complementary divisor of the congruence condition $\ell_1n_1 \equiv \ell_2n_2 \pmod m$ (with $\ell_1n_1 \neq \ell_2n_2$), so we write this as

$$md_0 = \ell_1\mathfrak{p}_1\mathfrak{p}_2n_1' - \ell_2\mathfrak{q}_1\mathfrak{q}_2n_2', \qquad d_0 \in \mathbb{Z}_{\neq 0}$$

with $\mathfrak{p}_i, \mathfrak{q}_i$ as in (4.5). We simplify the common factors and rewrite this equality as

$$md = \tilde{\ell}_1\mathfrak{p}_1n_1' - \tilde{\ell}_2\mathfrak{q}_2n_2' \tag{4.7}$$

for $d := d_0/\mathfrak{q}_1\mathfrak{p}_2$ and

$$\tilde{\ell}_1 := \frac{\ell_1}{\mathfrak{q}_1}, \quad \tilde{\ell}_2 := \frac{\ell_2}{\mathfrak{p}_2}. \tag{4.8}$$

We notice that the condition $(m, b\ell_1\ell_2n_1n_2) = 1$ can be factored into $(m, b\mathfrak{q}_1\mathfrak{p}_2) = 1$ and $(m, \tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2n_1'n_2') = 1$. Moreover, for $(\ell_1n_1', \ell_2n_2') = 1$, the conditions $(m, \tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2n_1'n_2') = 1$, $\eta|m$ and (4.7) can be expressed in the equivalent form

$$\begin{aligned} &(d, \tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2n_1'n_2') = 1, && \tilde{\ell}_1\mathfrak{p}_1n_1' \equiv \tilde{\ell}_2\mathfrak{q}_2n_2' \pmod{|d|\eta}, \\ &\mathfrak{q}_1\mathfrak{p}_2|d| \leq D := 3\frac{NL}{M}, && n_2' \in \mathcal{I}, \end{aligned}$$

for a certain interval $\mathcal{I} := \mathcal{I}(\ell_1, \ell_2, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, d, n_2')$. Thus, to eliminate the variable m, it remains to express the condition $(m, b\mathfrak{q}_1\mathfrak{p}_2) = 1$ and the argument of the exponential in terms of the remaining variables.

We do this by dividing the sum over m according to the residue classes $m \equiv c \pmod{b\mathfrak{q}_1\mathfrak{p}_2}$ for $c \in (\mathbb{Z}/b\mathfrak{q}_1\mathfrak{p}_2\mathbb{Z})^*$. Thus, using also (4.7), we obtain that m satisfies the following congruence conditions

$$m \equiv \overline{d}\tilde{\ell}_1\mathfrak{p}_1n_1' \pmod{\mathfrak{q}_2n_2'}, \ \ m \equiv -\overline{d}\tilde{\ell}_2\mathfrak{q}_2n_2' \pmod{\mathfrak{p}_1n_1'}, \ \ m \equiv c \pmod{b\mathfrak{q}_1\mathfrak{p}_2}$$

and the argument of the exponential function

$$\vartheta\left(\frac{a_1\overline{m}}{bn_1}-\frac{a_2\overline{m}}{bn_2}\right)=\vartheta\left(\frac{a_1\overline{m}}{b\mathfrak{p}_1\mathfrak{p}_2n_1'}-\frac{a_2\overline{m}}{b\mathfrak{q}_1\mathfrak{q}_2n_2'}\right)$$

is congruent modulo 1 to

$$\begin{aligned}\vartheta\Bigg(-d\frac{a_1\overline{b\mathfrak{p}_2\tilde{\ell}_2\mathfrak{q}_2n_2'}}{\mathfrak{p}_1n_1'}-d\frac{a_2\overline{\tilde{\ell}_1\mathfrak{p}_1n_1'b\mathfrak{q}_1}}{\mathfrak{q}_2n_2'}+\frac{a_1\overline{c\mathfrak{p}_1n_1'}}{b\mathfrak{p}_2}-\frac{a_2\overline{c\mathfrak{q}_2n_2'}}{b\mathfrak{q}_1}\Bigg)&\\ \equiv\vartheta\Bigg(\overline{c}\bigg(\frac{a_1\overline{\mathfrak{p}_1n_1'}}{b\mathfrak{p}_2}-\frac{a_2\overline{\mathfrak{q}_2n_2'}}{b\mathfrak{q}_1}\bigg)-d\bigg(\frac{a_1\overline{b\mathfrak{p}_2\tilde{\ell}_2\mathfrak{q}_2n_2'}}{\mathfrak{p}_1n_1'}+\frac{a_2\overline{\tilde{\ell}_1\mathfrak{p}_1n_1'b\mathfrak{q}_1}}{\mathfrak{q}_2n_2'}\bigg)\Bigg),&\quad(4.9)\end{aligned}$$

since for $(\beta,\gamma)=(\alpha,\beta\gamma)=1$, we have $\frac{\overline{\alpha}}{\beta\gamma}\equiv\frac{\overline{\alpha\beta}}{\gamma}+\frac{\overline{\alpha\gamma}}{\beta}\pmod 1$. Finally, the conditions $m\equiv c \pmod{b\mathfrak{q}_1\mathfrak{p}_2}$ and $\tilde{\ell}_1\mathfrak{p}_1n_1'\equiv\tilde{\ell}_2\mathfrak{q}_2n_2' \pmod{|d|\eta}$ can be combined into the equivalent

$$\tilde{\ell}_1\mathfrak{p}_1n_1'\equiv\tilde{\ell}_2\mathfrak{q}_2n_2'-c\eta d \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d|\eta}.$$

Thus,

$$\mathscr{S}_{b,\eta}=\sum_{\substack{\ell_1,\ell_2\in\mathcal{L},n_1,n_2\in\mathcal{N},a_1,a_2\in\mathcal{A},d\le D/\mathfrak{q}_1\mathfrak{p}_2\\(b\vartheta,\ell_1\ell_2n_1n_2)=(\ell_1,\ell_2)=(n_1',n_2')=1,\\(d,\tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2n_1'n_2')=(n_1'n_2',\ell_1\ell_2)=1,\ n_2'\in\mathcal{I},\\\tilde{\ell}_1\mathfrak{p}_1n_1'\neq\tilde{\ell}_2\mathfrak{q}_2n_2'}}\sum\sum\sum\sum\sum\sum\ \sideset{}{^*}\sum_{\substack{c\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2),\\\tilde{\ell}_1\mathfrak{p}_1n_1'\equiv\tilde{\ell}_2\mathfrak{q}_2n_2'-cd\eta\\(\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2|d|\eta)}}\beta_{n_1}\nu_{a_1}\overline{\beta_{n_2}\nu_{a_2}}\,\mathrm{e}(\cdots),$$

where the argument of the exponential is given by (4.9) and $n_i',\mathfrak{p}_i,\mathfrak{q}_i,\tilde{\ell}_i$ are defined in (4.5) and (4.8). Now, we treat $\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2,n_1',n_2'$ as variables and, after switching the order of summation, we have

$$\begin{aligned}\mathscr{S}_{b,\eta}=&\sum\sum\sum\sum_{\substack{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2\in\mathcal{L}\cup\{1\},\\(b\vartheta\mathfrak{p}_1\mathfrak{q}_1,\mathfrak{p}_2\mathfrak{q}_2)=1,\\(b\vartheta,\mathfrak{p}_1q_1)=1}}\ \sum\sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2n_1',\mathfrak{q}_1\mathfrak{q}_2n_2'\in\mathcal{N},\\(b\vartheta,n_1'n_2')=1,\\(n_1',n_2')=1}}\ \sum\sum_{\substack{\ell_1,\ell_2\in\mathcal{L},\\\mathfrak{p}_1,\mathfrak{q}_1|\ell_1,\ \mathfrak{p}_2,\mathfrak{q}_2|\ell_2,\\(bn_1'n_2'\vartheta,\ell_1\ell_2)=(\ell_1,\ell_2)=1,\\\tilde{\ell}_1\mathfrak{p}_1n_1'\neq\tilde{\ell}_2\mathfrak{q}_2n_2'}}\ \sum_{\substack{0\neq|d|\le D/\mathfrak{q}_1\mathfrak{p}_2,\\(d,\tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2n_1'n_2')=1\\n_2'\in\mathcal{I}}}\\&\sideset{}{^*}\sum_{\substack{c\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2),\\\tilde{\ell}_1\mathfrak{p}_1n_1'\equiv\tilde{\ell}_2\mathfrak{q}_2n_2'-cd\eta\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2|d|\eta)}}\ \sum\sum_{a_1,a_2\in\mathcal{A}}\beta_{\mathfrak{p}_1\mathfrak{p}_2n_1'}\nu_{a_1}\overline{\beta_{\mathfrak{q}_1\mathfrak{q}_2n_2'}\nu_{a_2}}\,\mathrm{e}(\cdots),\end{aligned}$$

with the argument of the exponential still given by (4.9).

4.1.2 Applying the Cauchy-Schwarz inequality

Next, we apply the Cauchy-Schwarz inequality with respect to the sums over $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, n_1', n_2', c, a_2$. After squaring out, we get

$$\mathscr{S}_{b,\eta}^2 \ll M^\varepsilon \|\beta\|^4 \|\nu\|^2 \sum\sum\sum\sum_{\substack{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2 \in \mathcal{L}\cup\{1\},\\ \mathfrak{p}_1\neq\mathfrak{q}_1 \Rightarrow 1\in\{\mathfrak{p}_1,\mathfrak{q}_1\},\\ \mathfrak{p}_2\neq\mathfrak{q}_2 \Rightarrow 1\in\{\mathfrak{p}_2,\mathfrak{q}_2\}}} b\mathfrak{q}_1\mathfrak{p}_2 \mathscr{T}_{b,\eta}, \tag{4.10}$$

where $\mathscr{T}_{b,\eta}$ is defined as

$$\begin{aligned} \mathscr{T}_{b,\eta} := & \sum\sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2 n_1', \mathfrak{q}_1\mathfrak{q}_2 n_2' \in \mathcal{N}\\ (b, n_1'n_2') = (n_1', \vartheta n_2') = 1,\\ \mu^2(n_1')=1}} \sum\sum\sum\sum_{\substack{\ell_1,\ell_2,\ell_1',\ell_2' \in \mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1 | (\ell_1,\ell_1'),\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2')\\ (b\vartheta n_1'n_2', \ell_1\ell_2\ell_1'\ell_2') = (\ell_1,\ell_2) = (\ell_1',\ell_2')=1,\\ \tilde{\ell}_1\mathfrak{p}_1 n_1' \neq \tilde{\ell}_2\mathfrak{q}_2 n_2',\ \tilde{\ell}_1'\mathfrak{p}_1 n_1' \neq \tilde{\ell}_2'\mathfrak{q}_2 n_2'}} \sum\sum_{\substack{0\neq|d|,|d'|\leq D/\mathfrak{q}_1\mathfrak{p}_2,\\ (d, \tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2 n_1'n_2')=1\\ (d', \tilde{\ell}_1'\tilde{\ell}_2'\mathfrak{p}_1\mathfrak{q}_2 n_1'n_2')=1\\ n_2' \in \mathcal{I}\cap\mathcal{I}'}} \\ & \sideset{}{^*}\sum_{\substack{c \pmod{b\mathfrak{q}_1\mathfrak{p}_2},\\ \tilde{\ell}_1\mathfrak{p}_1 n_1' \equiv \tilde{\ell}_2\mathfrak{q}_2 n_2' - cd\eta \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d|\eta},\\ \tilde{\ell}_1'\mathfrak{p}_1 n_1' \equiv \tilde{\ell}_2'\mathfrak{q}_2 n_2' - cd'\eta \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d'|\eta}}} \sum\sum\sum_{a_1,a_1',a_2\in\mathcal{A}} \nu_{a_1}\overline{\nu_{a_1'}}\, \mathrm{e}(\cdots), \end{aligned} \tag{4.11}$$

with $\mathcal{I}' := \mathcal{I}(\ell_1', \ell_2', \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, d', n_2')$, $\tilde{\ell}_1' := \ell_1'/\mathfrak{q}_1, \tilde{\ell}_2' := \ell_2'/\mathfrak{p}_2$ and where we introduced the condition $\mu^2(n_1') = 1$, which was implicit in the previous formulae, and we dropped the condition $(n_2', \vartheta) = 1$. The argument of the exponential is now

$$\vartheta\left(\bar{c}(a_1 - a_1')\frac{\overline{\mathfrak{p}_1 n_1'}}{b\mathfrak{p}_2} - \frac{(da_1\tilde{\ell}_2' - d'a_1'\tilde{\ell}_2)\overline{\tilde{\ell}_2\tilde{\ell}_2' b\mathfrak{p}_2\mathfrak{q}_2 n_2'}}{\mathfrak{p}_1 n_1'} - \frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)\overline{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1 n_1' b\mathfrak{q}_1}}{\mathfrak{q}_2 n_2'}\right). \tag{4.12}$$

We divide the right hand side of (4.11) into two parts:

$$\mathscr{T}_{b,\eta} = \mathscr{U}_{b,\eta} + \mathscr{U}_{b,\eta}^*, \tag{4.13}$$

where $\mathscr{U}_{b,\eta}$ is the contribution of the terms with $(\ell_1\ell_1', \ell_2\ell_2') = 1$. We will bound $\mathscr{U}_{b,\eta}$ in Section 4.1.3, and in Section 4.1.4 we will explain the modifications needed to bound $\mathscr{U}_{b,\eta}^*$.

4.1.3 Bounding $\mathscr{U}_{b,\eta}$: the case $(\ell_1\ell_1', \ell_2\ell_2') = 1$

We start by dividing $\mathscr{U}_{b,\eta}(M, N, A, \mathcal{L}, \nu)$ into

$$\mathscr{U}_{b,\eta} = \mathscr{V}_{b,\eta} + \mathscr{V}_{b,\eta}^*, \tag{4.14}$$

where $\mathscr{V}_{b,\eta}^*$ is the contribution of the terms such that $d = d', \ell_1 = \ell_1', \ell_2 = \ell_2'$ and $a_1 \neq a_1'$.

4.1.3.1 Bounding $\mathscr{V}_{b,\eta}^*$: the case $d = d', \ell_1 = \ell_1', \ell_2 = \ell_2', a_1 \neq a_1'$

In this section we deal with $\mathscr{V}_{b,\eta}^*$, which is given by

$$\mathscr{V}_{b,\eta}^* := \sum_{\substack{p_1\mathfrak{p}_2 n_1', \mathfrak{q}_1\mathfrak{q}_2 n_2' \in \mathcal{N} \\ (b, n_1'n_2') = (n_1', \vartheta n_2') = 1, \\ \mu^2(n_1') = 1}} \sum \sum_{\substack{\ell_1, \ell_2 \in \mathcal{L}, \\ \mathfrak{p}_1, \mathfrak{q}_1 | \ell_1,\ \mathfrak{p}_2, \mathfrak{q}_2 | \ell_2 \\ (b\vartheta n_1'n_2', \ell_1\ell_2) = (\ell_1, \ell_2) = 1, \\ \tilde{\ell}_1\mathfrak{p}_1 n_1' \neq \tilde{\ell}_2\mathfrak{q}_2 n_2'}} \sum \sum_{\substack{0 \neq |d| \leq D/\mathfrak{q}_1\mathfrak{p}_2, \\ (d, \tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2 n_1'n_2') = 1, \\ n_2' \in \mathcal{I}}} \sum_{\substack{c \pmod{b\mathfrak{q}_1\mathfrak{p}_2}, \\ \tilde{\ell}_1\mathfrak{p}_1 n_1' \equiv \tilde{\ell}_2\mathfrak{q}_2 n_2' - cd\eta \\ \pmod{b\mathfrak{q}_1\mathfrak{p}_2 |d|\eta}}}{}^{*}$$

$$\sum \sum \sum_{\substack{a_1, a_1', a_2 \in \mathcal{A}, \\ a_1 \neq a_1'}} \nu_{a_1} \overline{\nu_{a_1'}}\, \mathrm{e}\left(\vartheta \left(\overline{c}(a_1 - a_1') \frac{\overline{\mathfrak{p}_1 n_1'}}{b\mathfrak{p}_2} - \frac{d(a_1 - a_1')\overline{\tilde{\ell}_2 b\mathfrak{p}_2\mathfrak{q}_2 n_2'}}{\mathfrak{p}_1 n_1'} \right) \right).$$

We reintroduce the complementary divisor in the congruence condition $\tilde{\ell}_1\mathfrak{p}_1 n_1' \equiv \tilde{\ell}_2\mathfrak{q}_2 n_2' - cd\eta \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d|\eta}$ and reverse the previous computations. We then arrive at

$$\mathscr{V}_{b,\eta}^* = \sum \sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2 n_1', \mathfrak{q}_1\mathfrak{q}_2 n_2' \in \mathcal{N} \\ (b, n_1'n_2') = 1, \\ (n_1', \vartheta n_2') = 1, \\ \mu^2(n_1') = 1}} \sum \sum_{\substack{\ell_1, \ell_2 \in \mathcal{L}, \\ \mathfrak{p}_1, \mathfrak{q}_1 | \ell_1,\ \mathfrak{p}_2, \mathfrak{q}_2 | \ell_2 \\ (b\vartheta n_1'n_2', \ell_1\ell_2) = (\ell_1, \ell_2) = 1, \\ \tilde{\ell}_1\mathfrak{p}_1 n_1' \neq \tilde{\ell}_2\mathfrak{q}_2 n_2'}} \sum \sum \sum \sum_{\substack{m \in \mathcal{M},\ a_1, a_1', a_2 \in \mathcal{A},\ \eta | m \\ (m, b\ell_1\ell_2 n_1'n_2') = 1,\ a_1 \neq a_1' \\ \tilde{\ell}_1\mathfrak{p}_1 n_1' \equiv \tilde{\ell}_2\mathfrak{q}_2 n_2' \pmod{m},}} \nu_{a_1} \overline{\nu_{a_1'}}\, \mathrm{e}\left(\vartheta \frac{(a_1 - a_1')\overline{m}}{b\mathfrak{p}_1\mathfrak{p}_2 n_1'} \right).$$

Now, we introduce once again the complementary divisor, but this time we get rid of the variable n_2':

$$\mathscr{V}_{b,\eta}^* = \sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2 n_1' \in \mathcal{N}, \\ (b\vartheta, n_1') = 1, \\ \mu(n_1')^2 = 1}} \sum \sum_{\substack{\ell_1, \ell_2 \in \mathcal{L}, \\ \mathfrak{p}_1, \mathfrak{q}_1 | \ell_1,\ \mathfrak{p}_2, \mathfrak{q}_2 | \ell_2 \\ (b\vartheta n_1', \ell_1\ell_2) = (\ell_1, \ell_2) = 1}} \sum_{\substack{0 \neq |d| \leq D/\mathfrak{q}_1\mathfrak{p}_2, \\ (d, \tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2 n_1') = 1}} \sum \sum \sum_{\substack{a_1, a_1', a_2 \in \mathcal{A}, \\ a_1 \neq a_1'}} \nu_{a_1} \overline{\nu_{a_1'}} F(\cdots),$$

where

$$F(\cdots) := \sum_{\substack{m\in\mathcal{M}\cap\mathcal{J},\ (m,b\ell_1\ell_2 n_1')=1,\\ m\equiv\overline{d}\tilde{\ell}_1\mathfrak{p}_1 n_1' \pmod{\tilde{\ell}_2\mathfrak{q}_2},\ m\equiv 0 \pmod{\eta},\\ (md-\tilde{\ell}_1\mathfrak{p}_1 n_1', b\mathfrak{q}_1\ell_2\mathfrak{q}_2)=\tilde{\ell}_2\mathfrak{q}_2}} \mathrm{e}\Big(\vartheta\frac{(a_1-a_1')\overline{m}}{b\mathfrak{p}_1\mathfrak{p}_2 n_1'}\Big),$$

for some interval $\mathcal{J} = \mathcal{J}(\ell_1, \ell_2, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, d, n_2')$. Notice that, in order not to lose the condition $(n_2', b\mathfrak{q}_1\mathfrak{p}_2) = 1$, we have to introduce the condition $(md - \tilde{\ell}_1\mathfrak{p}_1 n_1', b\mathfrak{q}_1\ell_2\mathfrak{q}_2) = \tilde{\ell}_2\mathfrak{q}_2$. We apply (A.2) with $\beta = 0$, χ the trivial character and

$$k := \eta\tilde{\ell}_2\mathfrak{q}_2, \quad \gamma := b\mathfrak{p}_1\mathfrak{p}_2 n_1', \quad \gamma_1 := \frac{\gamma}{h_1} = \frac{b\mathfrak{p}_1\mathfrak{p}_2 n_1'}{(\mathfrak{q}_2,\mathfrak{p}_2)(\eta, n_1')}, \quad c = b\mathfrak{q}_1\ell_2\mathfrak{q}_2$$
$$h := (k,\gamma) = (\eta, n_1')(\tilde{\ell}_2\mathfrak{q}_2, \mathfrak{p}_2) = (\eta, n_1')(\mathfrak{q}_2, \mathfrak{p}_2), \qquad h_1 := (k^\infty, \gamma) = h,$$

where we used that $(\eta, b\ell_1\ell_2) = (n_1', \ell_2) = 1$ and that γ/b is square-free. It follows that

$$F(\cdots) \ll (bN)^{\frac{1}{2}+\varepsilon} + (a_1 - a_1', bn_1'\mathfrak{p}_1\mathfrak{p}_2)^{\frac{1}{2}} \frac{M}{\tilde{\ell}_2(b\mathfrak{p}_1\mathfrak{p}_2\mathfrak{q}_2 n_1')^{\frac{1}{2}}},$$

since $(\vartheta, bn_1'\ell_1\ell_2) = 1$. Dealing with the GCD as in (3.3), we obtain

$$\begin{aligned}
\mathscr{V}_{b,\eta}^* &\ll \|\nu\|^2 \frac{A^2DM^\varepsilon}{\mathfrak{q}_1\mathfrak{p}_2} \sum_{\mathfrak{p}_1\mathfrak{p}_2 n_1'\in\mathcal{N}} \sum\sum_{\substack{\ell_1,\ell_2\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|\ell_1,\ \mathfrak{p}_2,\mathfrak{q}_2|\ell_2}} \bigg((bN)^{\frac{1}{2}} + \frac{M}{\tilde{\ell}_2(b\mathfrak{p}_1\mathfrak{p}_2\mathfrak{q}_2 n_1')^{\frac{1}{2}}}\bigg) \\
&\ll \|\nu\|^2 \frac{A^2DM^\varepsilon}{\mathfrak{q}_1\mathfrak{p}_2} \sum_{\mathfrak{p}_1\mathfrak{p}_2 n_1'\in\mathcal{N}} \frac{L^2}{(\mathfrak{p}_1+\mathfrak{q}_1)(\mathfrak{p}_2+\mathfrak{q}_2)} \bigg((bN)^{\frac{1}{2}} + \frac{M}{\tilde{\ell}_2(b\mathfrak{q}_2 N)^{\frac{1}{2}}}\bigg) \\
&\ll \frac{\|\nu\|^2 A^2 D L^2 N M^\varepsilon \Big(\frac{(bN)^{\frac{1}{2}}}{\mathfrak{p}_1\mathfrak{p}_2} + \frac{M}{\mathfrak{p}_1 L(b\mathfrak{q}_2 N)^{\frac{1}{2}}}\Big)}{\mathfrak{q}_1\mathfrak{p}_2(\mathfrak{p}_1+\mathfrak{q}_1)(\mathfrak{p}_2+\mathfrak{q}_2)} \ll \frac{\|\nu\|^2 b^{\frac{1}{2}} L^2 N^{\frac{3}{2}} M^\varepsilon D A^2}{\mathfrak{q}_1\mathfrak{p}_2(\mathfrak{p}_1+\mathfrak{q}_1)(\mathfrak{p}_2+\mathfrak{q}_2)},
\end{aligned} \tag{4.15}$$

where we could assume without loss of generality that $1 \le D = 3\frac{LN}{M}$, since otherwise the sum over d in the definition of $\mathscr{V}_{b,\eta}^*$ is empty.

4.1.3.2 Bounding $\mathscr{V}_{b,\eta}$: the case $(d,\ell_1,\ell_2)\neq(d',\ell_1',\ell_2')$ if $a_1\neq a_1'$

Here we deal with $\mathscr{V}_{b,\eta}$, which consists of the terms of $\mathscr{U}_{b,\eta}$ such that $(d,\ell_1,\ell_2)\neq(d',\ell_1',\ell_2')$ if $a_1\neq a_1'$ (we remark that here $(\cdot,\cdot,\cdot)$ indicates a triple and not a GCD). Specifically,

$$\mathscr{V}_{b,\eta}:=\sum\sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2n_1',\mathfrak{q}_1\mathfrak{q}_2n_2'\in\mathcal{N}\\(b,n_1'n_2')=(n_1',\vartheta n_2')=1,\\ \mu^2(n_1')=1}}\ \sum\sum\sum\sum_{\substack{\ell_1,\ell_2,\ell_1',\ell_2'\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|(\ell_1,\ell_1'),\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2')\\ (bn_1'n_2'\vartheta,\ell_1\ell_2\ell_1'\ell_2')=(\ell_1\ell_1',\ell_2\ell_2')=1,\\ \tilde{\ell}_1\mathfrak{p}_1n_1'\neq\tilde{\ell}_2\mathfrak{q}_2n_2',\ \tilde{\ell}_1'\mathfrak{p}_1n_1'\neq\tilde{\ell}_2'\mathfrak{q}_2n_2'}}\ \sum\sum_{\substack{0\neq|d|,|d'|\leq D/\mathfrak{q}_1\mathfrak{p}_2,\\ (d,\tilde{\ell}_1\tilde{\ell}_2\mathfrak{p}_1\mathfrak{q}_2n_1'n_2')=1\\ (d',\tilde{\ell}_1'\tilde{\ell}_2'\mathfrak{p}_1\mathfrak{q}_2n_1'n_2')=1\\ n_2'\in\mathcal{I}\cap\mathcal{I}'}}$$
$$\sum_{\substack{c\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2),\\ \tilde{\ell}_1\mathfrak{p}_1n_1'\equiv\tilde{\ell}_2\mathfrak{q}_2n_2'-cd\eta\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2|d|\eta),\\ \tilde{\ell}_1'\mathfrak{p}_1n_1'\equiv\tilde{\ell}_2'\mathfrak{q}_2n_2'-cd'\eta\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2|d'|\eta)}}^{*}\ \sum\sum\sum_{\substack{a_1,a_1',a_2\in\mathcal{A},\\ a_1\neq a_1'\Rightarrow(d,\ell_1,\ell_2)\neq(d',\ell_1',\ell_2')}}\nu_{a_1}\overline{\nu}_{a_1'}\,\mathrm{e}(\cdots),$$

with the argument of the exponential given by (4.12).

We start the analysis of $\mathscr{V}_{b,\eta}$ by noticing that the conditions

$$\tilde{\ell}_1\mathfrak{p}_1n_1'-\tilde{\ell}_2\mathfrak{q}_2n_2'+cd\eta\equiv0\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2|d|\eta)$$
$$\tilde{\ell}_1'\mathfrak{p}_1n_1'-\tilde{\ell}_2'\mathfrak{q}_2n_2'+cd'\eta\equiv0\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2|d'|\eta)$$

imply the congruence conditions

$$(\tilde{\ell}_2'\tilde{\ell}_1-\tilde{\ell}_2\tilde{\ell}_1')\mathfrak{p}_1n_1'+(\tilde{\ell}_2'd-\tilde{\ell}_2d')c\eta\equiv0\ (\mathrm{mod}\ b\mathfrak{q}_1\mathfrak{p}_2\eta),\qquad(4.16)$$
$$(d'\tilde{\ell}_1-d\tilde{\ell}_1')\mathfrak{p}_1n_1'\equiv(d'\tilde{\ell}_2-d\tilde{\ell}_2')\mathfrak{q}_2n_2'\ (\mathrm{mod}\ b)$$

and thus also

$$(d'\tilde{\ell}_1-d\tilde{\ell}_1')\overline{\mathfrak{q}_2n_2'}\equiv(d'\tilde{\ell}_2-d\tilde{\ell}_2')\overline{\mathfrak{p}_1n_1'}\ (\mathrm{mod}\ b),\qquad(4.17)$$

since $(b,\mathfrak{p}_1n_1'\mathfrak{q}_2n_2')=1$.

Now, we use the congruence relation (which is also in some sense an instance of divisor switching)

$$\frac{\overline{\alpha\gamma}}{\beta}+\frac{\overline{\beta\gamma}}{\alpha}+\frac{\overline{\alpha\beta}}{\gamma}\equiv\frac{1}{\alpha\beta\gamma}\ (\mathrm{mod}\ 1),\qquad(4.18)$$

which holds for α,β,γ pairwise coprime, to rewrite $-\frac{a_2(d\tilde{\ell}_1'-d'\tilde{\ell}_1)\overline{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1n_1'b\mathfrak{q}_1}}{\mathfrak{q}_2n_2'}$

(modulo 1) as

$$\begin{aligned}
&\frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)\overline{\mathfrak{q}_2 n_2' b}}{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1\mathfrak{q}_1 n_1'} + \frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)\overline{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1\mathfrak{q}_1 n_1'\mathfrak{q}_2 n_2'}}{b} - \frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)}{b\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1 n_1'\mathfrak{q}_1\mathfrak{q}_2 n_2'} \\
&\equiv \frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)\overline{\mathfrak{q}_2 n_2' b}}{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1\mathfrak{q}_1 n_1'} + \frac{a_2(d'\tilde{\ell}_2 - d\tilde{\ell}_2')\overline{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1^2\mathfrak{q}_1 n_1' n_1'^2}}{b} - \frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)}{b\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1 n_1'\mathfrak{q}_1\mathfrak{q}_2 n_2'},
\end{aligned} \tag{4.19}$$

by (4.17), and thus the argument (4.12) of the exponential becomes

$$\vartheta\left(\Delta\frac{\overline{\tilde{\ell}_2\tilde{\ell}_2' b\mathfrak{p}_2\mathfrak{q}_2 n_2'}}{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{q}_1\mathfrak{p}_1 n_1'} - \frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)}{b\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1 n_1'\mathfrak{q}_1\mathfrak{q}_2 n_2'} + (a_1 - a_1')\frac{\overline{c\mathfrak{p}_1 n_1'}}{b\mathfrak{p}_2} - \frac{a_2(d'\tilde{\ell}_2 - d\tilde{\ell}_2')\overline{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1^2\mathfrak{q}_1 n_1'^2}}{b}\right), \tag{4.20}$$

where

$$\Delta := a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)\tilde{\ell}_2\tilde{\ell}_2'\mathfrak{p}_2 - (da_1\tilde{\ell}_2' - d'a_1'\tilde{\ell}_2)\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{q}_1.$$

We divide $\mathscr{V}_{b,\eta}$ into two parts, depending on whether Δ is equal to 0 or not:

$$\mathscr{V}_{b,\eta} = \mathscr{V}_{b,\eta}^{\Delta=0} + \mathscr{V}_{b,\eta}^{\Delta\neq 0}. \tag{4.21}$$

We shall give a trivial bound for the terms with $\Delta = 0$, whereas we will use Weil's bound on the sum over n_2' to handle the terms with $\Delta \neq 0$.

The terms with $\Delta = 0$. The condition $\Delta = 0$ gives that

$$a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)\tilde{\ell}_2\tilde{\ell}_2'\mathfrak{p}_2 = (da_1\tilde{\ell}_2' - d'a_1'\tilde{\ell}_2)\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{q}_1, \tag{4.22}$$

and it allows us to express either a_1 or a_1' in terms of all the other variables:

$$\begin{aligned}
a_1 &= f(a_1', a_2, d, d', \ell_1, \ell_2, \ell_1', \ell_2', \mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{p}_2, \mathfrak{q}_2), \\
a_1' &= g(a_1, a_2, d, d', \ell_1, \ell_2, \ell_1', \ell_2', \mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{p}_2, \mathfrak{q}_2),
\end{aligned}$$

for some functions f and g. Moreover, since $(\ell_1\ell_1', \ell_2\ell_2') = (d, \tilde{\ell}_1\tilde{\ell}_2) = (d', \tilde{\ell}_1'\tilde{\ell}_2') = 1$, then (4.22) implies that $\tilde{\ell}_1 | a_2\tilde{\ell}_1'$, $\tilde{\ell}_1' | a_2\tilde{\ell}_1$ and $\tilde{\ell}_2 | a_1\tilde{\ell}_2'$, $\tilde{\ell}_2' | a_1'\tilde{\ell}_2$, which, by definition of $\tilde\ell_i, \tilde\ell_i'$, is equivalent to $\ell_1|a_2\ell_1'$, $\ell_1'|a_2\ell_1$, $\ell_2|a_1\ell_2'$, $\ell_2'|a_1'\ell_2$. It follows that $\mathscr{V}_{b,\eta}^{\Delta=0}$ is bounded by

$$\begin{aligned}\mathscr{V}_{b,\eta}^{\Delta=0} \ll & \sum\sum_{\mathfrak{p}_1\mathfrak{p}_2 n_1',\mathfrak{q}_1\mathfrak{q}_2 n_2'\in\mathcal{N}} \sum\sum\sum\sum_{\substack{\ell_1,\ell_2,\ell_1',\ell_2'\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|(\ell_1,\ell_1'),\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2'),\\ \tilde\ell_1\mathfrak{p}_1 n_1'\neq\tilde\ell_2\mathfrak{q}_2 n_2',\ \tilde\ell_1'\mathfrak{p}_1 n_1'\neq\tilde\ell_2'\mathfrak{q}_2 n_2'}} \sum\sum_{\substack{0\neq|d|,|d'|\leq\frac{D}{\mathfrak{q}_1\mathfrak{p}_2},\\ d'|(\tilde\ell_1'\mathfrak{p}_1 n_1'-\tilde\ell_2'\mathfrak{q}_2 n_2')}} \\ & \sum_{\substack{c \pmod{b\mathfrak{q}_1\mathfrak{p}_2},\\ \tilde\ell_1\mathfrak{p}_1 n_1'\equiv\tilde\ell_2\mathfrak{q}_2 n_2'-cd\eta,\\ \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d|\eta}}} \sum\sum\sum_{\substack{a_1,a_1',a_2\in\mathcal{A},\\ \ell_1|a_2\ell_1',\ell_1'|a_2\ell_1,\ell_2|a_1\ell_2',\ell_2'|a_1'\ell_2,\\ a_1=f(\cdots),\ a_1'=g(\cdots)}} (|\nu_{a_1}|^2+|\nu_{a_1'}|^2) \\ \ll & M^\varepsilon \sum_{\mathfrak{p}_1\mathfrak{p}_2 n_1',\mathfrak{q}_1\mathfrak{q}_2 n_2'\in\mathcal{N}} \sum\sum\sum\sum_{\substack{\ell_1,\ell_2,\ell_1',\ell_2'\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|(\ell_1,\ell_1'),\\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2')}} \sum\sum_{\substack{a_1,a_2\in\mathcal{A},\\ \ell_1|a_2\ell_1',\ \ell_1'|a_2\ell_1,\\ \ell_2|a_1\ell_2'}} |\nu_{a_1}|^2 \\ \ll & \|\nu\|^2\frac{AN^2M^\varepsilon}{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{q}_1\mathfrak{q}_2} \sum\sum_{\substack{\ell_1',\ell_2'\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|\ell_1',\ \mathfrak{p}_2,\mathfrak{q}_2|\ell_2'}} 1 \ll \frac{\|\nu\|^2 AL^2N^2M^\varepsilon}{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{q}_1\mathfrak{q}_2(\mathfrak{p}_1+\mathfrak{q}_1)(\mathfrak{p}_2+\mathfrak{q}_2)}, \end{aligned} \tag{4.23}$$

where we dropped the condition $(c, b\mathfrak{q}_1\mathfrak{p}_2)=1$ by positivity, and the sum over c has the only effect of turning the congruence condition $\tilde\ell_1\mathfrak{p}_1 n_1'\equiv\tilde\ell_2\mathfrak{q}_2 n_2' - cd\eta \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d|\eta}$ into $\tilde\ell_1\mathfrak{p}_1 n_1'\equiv\tilde\ell_2\mathfrak{q}_2 n_2' \pmod{|d|\eta}$, which then implies $d|(\tilde\ell_1\mathfrak{p}_1 n_1' - \tilde\ell_2\mathfrak{q}_2 n_2')$.

The terms with $\Delta\neq 0$. Here we bound $\mathscr{V}_{b,\eta}^{\Delta\neq0}$. Exchanging the order of summation and indicating with $G(\dots)$ the sum over n_2', we see that

$$\begin{aligned}\mathscr{V}_{b,\eta}^{\Delta\neq0} \ll & \sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2 n_1'\in\mathcal{N},\\ (b\vartheta,n_1')=1,\\ \mu^2(n_1')=1}} \sum\sum\sum\sum_{\substack{\ell_1,\ell_2,\ell_1',\ell_2'\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|(\ell_1,\ell_1'),\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2')\\ (bn_1'\vartheta,\ell_1\ell_2\ell_1'\ell_2')=(\ell_1\ell_1',\ell_2\ell_2')=1}} \sum\sum_{\substack{0\neq|d|,|d'|\leq D/\mathfrak{q}_1\mathfrak{p}_2,\\ (d,\tilde\ell_1\tilde\ell_2\mathfrak{p}_1\mathfrak{q}_2 n_1')=1\\ (d',\tilde\ell_1'\tilde\ell_2'\mathfrak{p}_1\mathfrak{q}_2 n_1')=1}} \\ & \sideset{}{^*}\sum_{\substack{c \pmod{b\mathfrak{q}_1\mathfrak{p}_2},\\ (\tilde\ell_2'\tilde\ell_1-\tilde\ell_2\tilde\ell_1')\mathfrak{p}_1 n_1'\equiv(\tilde\ell_2 d'-\tilde\ell_2' d)c\eta\\ \pmod{b\mathfrak{q}_1\mathfrak{p}_2\eta}}} \sum\sum\sum_{\substack{a_1,a_1',a_2\in\mathcal{A},\\ \Delta\neq0,\\ (d,\ell_1,\ell_2)\neq(d',\ell_1',\ell_2')}} |\nu_{a_1}\nu_{a_1'}||G(\cdots)|, \end{aligned} \tag{4.24}$$

where

$$G(\cdots) := \sum_{n_2',(*)} \mathrm{e}(\cdots), \tag{4.25}$$

and the argument of the exponential is

$$\vartheta\Delta\frac{\overline{\tilde{\ell}_2\tilde{\ell}_2' b\mathfrak{p}_2\mathfrak{q}_2 n_2'}}{\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{q}_1\mathfrak{p}_1 n_1'} - \frac{\vartheta a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)}{b\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1 n_1' b\mathfrak{q}_1\mathfrak{q}_2 n_2'} \pmod 1. \tag{4.26}$$

The condition $(*)$ indicates that n_2' satisfies the following conditions:

$$\begin{aligned} &n_2' \in \mathcal{I}\cap\mathcal{I}', \quad \mathfrak{q}_1\mathfrak{q}_2 n_2' \in \mathcal{N}, && \tilde{\ell}_1\mathfrak{p}_1 n_1' \equiv \tilde{\ell}_2\mathfrak{q}_2 n_2' - cd\eta \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d|\eta},\\ &(n_2', bdd'n_1'\ell_1\ell_2\ell_1'\ell_2') = 1, && \tilde{\ell}_1'\mathfrak{p}_1 n_1' \equiv \tilde{\ell}_2'\mathfrak{q}_2 n_2' - cd'\eta \pmod{b\mathfrak{q}_1\mathfrak{p}_2|d'|\eta},\\ &\tilde{\ell}_1\mathfrak{p}_1 n_1' \neq \tilde{\ell}_2\mathfrak{q}_2 n_2', && \tilde{\ell}_1'\mathfrak{p}_1 n_1' \neq \tilde{\ell}_2'\mathfrak{q}_2 n_2'. \end{aligned}$$

We remark that we could keep the condition (4.16) and drop the last two summands of (4.20) as they do not depend on n_2'. Also, notice that the condition $(d,\ell_1,\ell_2) \neq (d',\ell_1',\ell_2')$ if $a_1 \neq a_1'$ becomes simply $(d,\ell_1,\ell_2) \neq (d',\ell_1',\ell_2')$ since if these triples are equal then $\Delta \neq 0$ implies $a_1 \neq a_1'$.

We apply Lemma 3 to the sum over n_2', removing the second summand of (4.26) by using partial summation with the bound

$$\vartheta\frac{a_2(d\tilde{\ell}_1' - d'\tilde{\ell}_1)}{b\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{p}_1 n_1'\mathfrak{q}_1\mathfrak{q}_2 n_2'} \ll \frac{|\vartheta|ADL}{b\tilde{\ell}_1\tilde{\ell}_1'\mathfrak{q}_1^2\mathfrak{p}_1\mathfrak{p}_2 n_1'\mathfrak{q}_1\mathfrak{q}_2 n_2'} \ll \frac{|\vartheta|AD}{bLN^2}.$$

Note that n_2' runs over a finite union of intervals of length at most $O(N/\mathfrak{q}_1\mathfrak{q}_2)$, with a congruence condition modulo $b\mathfrak{q}_1\mathfrak{p}_2[d,d']\eta$ (provided that the sum is non-empty), where $[a,b]$ is the LCM of a and b.

In the notation of Lemma 3 (and under the various conditions of the sums in (4.24)), we have $\gamma = \tilde{\ell}_1\tilde{\ell}_1'\mathfrak{q}_1\mathfrak{p}_1 n_1'$, $k = b\mathfrak{q}_1\mathfrak{p}_2[d,d']\eta$ and

$$\begin{aligned} h &= (\gamma,k) = \mathfrak{q}_1([d,d'],\tilde{\ell}_1\tilde{\ell}_1')(\eta,n_1') = \mathfrak{q}_1(d,\tilde{\ell}_1')(d',\tilde{\ell}_1)(\eta,n_1'),\\ h_1 &= (k^\infty,\gamma) = (\mathfrak{q}_1^\infty[d,d']^\infty,\tilde{\ell}_1\tilde{\ell}_1')(\eta,n_1')\mathfrak{q}_1(\mathfrak{p}_1,\mathfrak{q}_1) = (\mathfrak{p}_1,\mathfrak{q}_1)h, \end{aligned}$$

since $\mathfrak{q}_1 > 1$ implies $\tilde{\ell}_1 = \tilde{\ell}_1' = 1$, whence

$$\gamma_1 := \frac{\gamma}{h_1} = \frac{\mathfrak{p}_1}{(\mathfrak{p}_1,\mathfrak{q}_1)}\frac{n_1'}{(\eta,n_1')}\frac{\tilde{\ell}_1}{(d',\tilde{\ell}_1)}\frac{\tilde{\ell}_1'}{(d,\tilde{\ell}_1')}.$$

Thus, Lemma 3 gives

$$G(\cdots) \ll M^\varepsilon\bigg((\mathfrak{p}_1,\mathfrak{q}_1)\Big(\frac{N\tilde{\ell}_1\tilde{\ell}_1'}{(\mathfrak{p}_1,\mathfrak{q}_1)\mathfrak{p}_2}\Big)^{\frac{1}{2}} + \frac{(\vartheta\Delta,\gamma_1)}{\gamma_1}\frac{N}{b\eta\mathfrak{q}_1^2\mathfrak{q}_2\mathfrak{p}_2[d,d']}\bigg)\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big)$$

$$\ll M^{\varepsilon}\Big(LN^{\frac12}+\frac{(\Delta,n_1')\mathfrak{p}_1}{b\mathfrak{q}_1^2\mathfrak{q}_2[d,d']}\Big)\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big).$$

Now, if we define

$$u:=(\tilde\ell_2'd-\tilde\ell_2d',b\mathfrak{q}_1\mathfrak{p}_2),\qquad v:=(\tilde\ell_2'd-\tilde\ell_2d')/u$$

(so that $(v,b\mathfrak{q}_1\mathfrak{p}_2/u)=1$), then by the congruence condition (4.16) we have

$$\tilde\ell_2'd\equiv\tilde\ell_2d' \pmod u, \tag{4.27}$$

$$\mathfrak{p}_1\tilde\ell_1\equiv\mathfrak{p}_1\overline{\tilde\ell_2'}\tilde\ell_2\tilde\ell_1' \pmod u \tag{4.28}$$

since $(\tilde\ell_2',b\mathfrak{q}_1\mathfrak{p}_2)=1$. Also, if $v\neq0$, then

$$c\equiv-\overline{v\eta}\frac{(\tilde\ell_2'\tilde\ell_1-\tilde\ell_2\tilde\ell_1')\mathfrak{p}_1n_1'}{u}\ \Big(\mathrm{mod}\ \frac{b\mathfrak{q}_1\mathfrak{p}_2}{u}\Big). \tag{4.29}$$

We remark that we can assume $u\leq\frac{10DL}{\mathfrak{q}_1\mathfrak{p}_2}$. Indeed, if $u>\frac{10DL}{\mathfrak{q}_1\mathfrak{p}_2}>\tilde\ell_2'd+\tilde\ell_2d'$ then (4.27) implies $\tilde\ell_2'd=\tilde\ell_2d'$. Thus $u=b\mathfrak{q}_1\mathfrak{p}_2$ and $d=d'$, $\ell_2=\ell_2'$ since $(d,\tilde\ell_2)=(d',\tilde\ell_2')=1$. Moreover, (4.28) would give $\mathfrak{p}_1\tilde\ell_1=\mathfrak{p}_1\tilde\ell_1'$ (since we can assume $D/\mathfrak{q}_1\mathfrak{p}_2\geq1$) and thus $\ell_1=\ell_1'$. So $(d,\ell_1,\ell_2)=(d',\ell_1',\ell_2')$, and these terms have been previously excluded.

Thus, if we bound trivially the sum over c using (4.29) and drop some conditions by positivity, we find

$$\mathscr{V}_{b,\eta}^{\Delta\neq0}\ll M^{\varepsilon}\sum_{\mathfrak{p}_1\mathfrak{p}_2n_1'\in\mathcal{N}}\ \sum\sum\sum\sum_{\substack{\ell_1,\ell_2,\ell_1',\ell_2'\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|(\ell_1,\ell_1'),\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2'),\\ (\ell_2',b\mathfrak{q}_1\mathfrak{p}_2)=1}}\ \sum_{\substack{u|b\mathfrak{q}_1\mathfrak{p}_2,\ u\leq\frac{10DL}{\mathfrak{q}_1\mathfrak{p}_2},\\ \mathfrak{p}_1\tilde\ell_1\equiv\mathfrak{p}_1\overline{\tilde\ell_2'}\tilde\ell_2\tilde\ell_1'\ (\mathrm{mod}\ u)}}\ \sum\sum_{\substack{|d|,|d'|\leq D/\mathfrak{q}_1\mathfrak{p}_2,\\ d\equiv\overline{\tilde\ell_2'}\tilde\ell_2d'\ (\mathrm{mod}\ u)}}$$

$$\sum\sum\sum_{\substack{a_1,a_1',a_2\in\mathcal{A},\\ \Delta\neq0}}u\,|\nu_{a_1}\nu_{a_1'}|\Big(LN^{\frac12}+\frac{(\Delta,n_1')\mathfrak{p}_1}{b\mathfrak{q}_1^2\mathfrak{q}_2[d,d']}\Big)\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big)$$

$$\ll\|\nu\|^2\frac{A^2LN^{\frac32}M^{\varepsilon}}{\mathfrak{p}_1\mathfrak{p}_2}\sum\sum\sum\sum_{\substack{\ell_1,\ell_2,\ell_1',\ell_2'\in\mathcal{L},\\ \mathfrak{p}_1,\mathfrak{q}_1|(\ell_1,\ell_1'),\\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2'),\\ (\ell_2',b\mathfrak{q}_1\mathfrak{p}_2)=1}}\ \sum_{\substack{u|b\mathfrak{q}_1\mathfrak{p}_2,\ u\leq\frac{10DL}{\mathfrak{q}_1\mathfrak{p}_2},\\ \mathfrak{p}_1\tilde\ell_1\equiv\mathfrak{p}_1\overline{\tilde\ell_2'}\tilde\ell_2\tilde\ell_1'\\ (\mathrm{mod}\ u)}}\ \sum\sum_{\substack{|d|,|d'|\leq D/\mathfrak{q}_1\mathfrak{p}_2,\\ d\equiv\overline{\tilde\ell_2'}\tilde\ell_2d'\ (\mathrm{mod}\ u)}}u\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big)$$

$$\ll\|\nu\|^2\frac{A^2DL^2N^{\frac32}M^{\varepsilon}}{(\mathfrak{p}_1+\mathfrak{q}_1)\mathfrak{p}_1\mathfrak{q}_1\mathfrak{p}_2^2}\sum\sum_{\substack{\ell_2,\ell_2'\in\mathcal{L},\\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2')}}\ \sum_{\substack{u|b\mathfrak{q}_1\mathfrak{p}_2,\\ u\leq\frac{10DL}{\mathfrak{q}_1\mathfrak{p}_2}}}u\Big(\frac{L}{u}+1\Big)\Big(\frac{D}{u\mathfrak{q}_1\mathfrak{p}_2}+1\Big)\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big)$$

$$\ll \|\nu\|^2 \frac{A^2 D L^2 N^{\frac{3}{2}} M^{\varepsilon}}{(\mathfrak{p}_1+\mathfrak{q}_1)\mathfrak{p}_1\mathfrak{q}_1\mathfrak{p}_2^2} \sum_{\substack{\ell_2,\ell_2'\in\mathcal{L},\\ \mathfrak{p}_2,\mathfrak{q}_2|(\ell_2,\ell_2')}}\sum \frac{DL}{\mathfrak{q}_1\mathfrak{p}_2}\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big)$$

$$\ll \frac{\|\nu\|^2 A^2 D^2 L^5 N^{\frac{3}{2}} M^{\varepsilon}}{(\mathfrak{p}_1+\mathfrak{q}_1)(\mathfrak{p}_2+\mathfrak{q}_2)\mathfrak{p}_1\mathfrak{q}_1^2\mathfrak{p}_2^3\mathfrak{q}_2}\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big). \tag{4.30}$$

4.1.3.3 Total bound for the terms with $(\ell_1\ell_1', \ell_2\ell_2') = 1$

By (4.21), (4.23) and (4.30), we obtain

$$\mathscr{V}_{b,\eta} \ll \|\nu\|^2 M^{\varepsilon}\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big)\frac{A^2D^2L^5N^{\frac{3}{2}}+AL^2N^2}{\mathfrak{q}_1\mathfrak{p}_2(\mathfrak{p}_1+\mathfrak{q}_1)(\mathfrak{p}_2+\mathfrak{q}_2)}$$

and thus, by (4.14) and (4.15), we have

$$\mathscr{U}_{b,\eta} \ll \|\nu\|^2 A^2 L^2 N^{\frac{3}{2}} M^{\varepsilon}\frac{(Db^{\frac{1}{2}}+L^3D^2+N^{\frac{1}{2}}/A)}{\mathfrak{q}_1\mathfrak{p}_2(\mathfrak{p}_1+\mathfrak{q}_1)(\mathfrak{p}_2+\mathfrak{q}_2)}\Big(1+\frac{|\vartheta|AD}{bLN^2}\Big). \tag{4.31}$$

4.1.4 Bounding $\mathscr{U}_{b,\eta}^*$: the case $(\ell_1\ell_1', \ell_2\ell_2') > 1$

First, we observe that if $(\ell_1\ell_1', \ell_2\ell_2') > 1$, then $(\ell_2, \ell_2') = (\ell_1, \ell_1') = 1$ since we have $(\ell_1, \ell_2) = (\ell_1', \ell_2') = 1$, and so $\mathfrak{p}_1 = \mathfrak{q}_1 = \mathfrak{p}_2 = \mathfrak{q}_2 = 1$. Thus, we can repeat the same arguments of Section 4.1.3 in a slightly different but simplified form, and we obtain that the bound (4.31) holds also for $\mathscr{U}_{b,\eta}^*$.

The only difference between this case and the $(\ell_1\ell_1', \ell_2\ell_2') = 1$ case is that we cannot make the same choice of α,β,γ in (4.18) as α,β,γ would not be pairwise coprime (and neither we could invert $\tilde{\ell}_2\tilde{\ell}_2' \pmod{\tilde{\ell}_1\tilde{\ell}_1'}$ in (4.20)). To overcome this problem, it is enough to divide the sum over n_2' into congruence classes modulo $(\ell_1\ell_1', \ell_2\ell_2')$ and apply (4.18) with $\gamma = b(\ell_1\ell_1', \ell_2\ell_2')$ rather than $\gamma = b$ as in (4.19). The extra sum over the congruence classes modulo $(\ell_1\ell_1', \ell_2\ell_2')$ has the effect of making us lose a factor of $(\ell_1\ell_1', \ell_2\ell_2')$ in the $\Delta \neq 0$ terms, but this loss is recovered by the extra condition between the $\ell_1, \ell_1', \ell_2, \ell_2'$. In fact, one can obtain a bound stronger than (4.31) in this case, since the resulting Kloosterman has smaller modulus.

4.1.5 The final bound for $\mathscr{S}_{b,\eta}$

Putting together (4.10), (4.13) with the bound (4.31) and its analogue for $\mathscr{U}^*_{b,\eta}$, we obtain

$$\begin{aligned}\mathscr{S}_{b,\eta} &\ll \|\beta\|^2\|\nu\|^2\Big(b+\frac{|\vartheta|AD}{LN^2}\Big)^{\frac12}ALN^{\frac34}M^{\varepsilon}\Big(D^{\frac12}b^{\frac14}+L^{\frac32}D+\frac{N^{\frac14}}{A^{\frac12}}\Big)\\ &\ll \|\beta\|^2\|\nu\|^2\Big(b+\frac{|\vartheta|A}{NM}\Big)^{\frac12}ALN^{\frac34}M^{\varepsilon}\Big(\frac{b^{\frac14}N^{\frac12}L^{\frac12}}{M^{\frac12}}+\frac{L^{\frac52}N}{M}+\frac{N^{\frac14}}{A^{\frac12}}\Big),\end{aligned}\tag{4.32}$$

since $D=3\frac{NL}{M}$.

4.2 The terms with $(n_1',n_2')>1$

In this section we bound $\mathscr{S}^*_{b,\eta}(M,N,A,L;\beta,\nu)$, which consists of the sum (4.2) restricted to $(n_1',n_2')>1$. We recall the definition (4.5) of n_1', n_2':

$$n_1':=\frac{n_1}{(n_1,\ell_1\ell_2)},\quad n_2':=\frac{n_2}{(n_2,\ell_1\ell_2)}$$

We write $\mu=(n_1',n_2')$. This implies $(\mu,\ell_1\ell_2)=1$ and $n_1=\mu h_1$, $n_2=\mu h_2$. Thus, denoting $h_1':=\frac{h_1}{(h_1,\ell_1,\ell_2)}$, $h_2':=\frac{h_2}{(h_2,\ell_1,\ell_2)}$, we automatically have $(h_1',h_2')=(h_1'h_2',\ell_1\ell_2)=1$ since n_1,n_2 are square-free. It follows that

$$\begin{aligned}\mathscr{S}^*_{b,\eta} &= \sum_{\substack{\mu>1,\\(\mu,\vartheta b)=1}}\ \sum\sum\sum\sum\sum\sum\sum_{\substack{\ell_1,\ell_2\in\mathcal{L},m\in\mathcal{M},h_1,h_2\in\mathcal{N}_\mu,a_1,a_2\in\mathcal{A}\\(mb\mu\vartheta,\ell_1\ell_2h_1h_2)=(m,\mu b)=(\ell_1,\ell_2)=(h_1',h_2')=1,\\ \ell_1h_1\equiv\ell_2h_2 \pmod m,\ (h_1'h_2',\ell_1\ell_2)=1\\ \ell_1h_1\neq\ell_2h_2,\ \eta|m}} \beta_{\mu h_1}\nu_{a_1}\overline{\beta_{\mu h_2}\nu_{a_2}}\,\mathrm{e}\Big(\frac{a_1\overline{m}}{b\mu h_1}-\frac{a_2\overline{m}}{b\mu h_2}\Big)\\ &= \sum_{\substack{1<\mu\le N,\\(\mu,\vartheta b)=1}}\mathscr{S}_{b,\eta}(M,N/\mu,A,\mathcal{L},\beta_\mu,\nu),\end{aligned}$$

where $\beta_\mu(n):=\beta_{\mu n}$ and $\mathcal{N}_\mu=[N/(2\mu),N/\mu]$. Thus, by (4.32)

$$\begin{aligned}\mathscr{S}^*_{b,\eta} &\ll \|\nu\|^2\Big(b+\frac{|\vartheta|A}{NM}\Big)^{\frac12}ALN^{\frac34}M^{\varepsilon}\sum_{\mu\le N}\frac{\|\beta_\mu\|^2}{\mu^{\frac14}}\Big(\frac{b^{\frac14}N^{\frac12}L^{\frac12}}{\mu^{\frac14}M^{\frac12}}+\frac{L^{\frac52}N}{\mu M}+\frac{N^{\frac14}}{\mu^{\frac14}A^{\frac12}}\Big)\\ &\ll \|\beta\|^2\|\nu\|^2\Big(b+\frac{|\vartheta|A}{NM}\Big)^{\frac12}ALN^{\frac34}M^{\varepsilon}\Big(\frac{b^{\frac14}N^{\frac12}L^{\frac12}}{M^{\frac12}}+\frac{L^{\frac52}N}{M}+\frac{N^{\frac14}}{A^{\frac12}}\Big).\end{aligned}\tag{4.33}$$

Combining (4.6) and the bounds (4.32) and (4.33), we obtain Lemma 2.

5 Optimizing the parameter L

Combining (2.3) with the bounds for the diagonal (3.1) and off-diagonal terms (4.4) we obtain

$$\mathcal{D}_b \ll \|\beta\|^2\|\nu\|^2\Big(1+\frac{|\vartheta|A}{bNM}\Big)^{\frac12} LM^{\varepsilon}\bigg(A(bLN)^{\frac12}+\frac{AM}{bN}+M + \frac{b^{\frac34}AN^{\frac54}L^{\frac12}}{M^{\frac12}}+\frac{b^{\frac12}AL^{\frac52}N^{\frac74}}{M}+b^{\frac12}A^{\frac12}N\bigg)$$

and thus, by (2.2),

$$\mathcal{C}_b \ll \|\beta\|^2\|\nu\|^2M^{\varepsilon}\Big(1+\frac{|\vartheta|A}{bNM}\Big)^{\frac12}\bigg(\frac{AM(bN)^{\frac12}}{L^{\frac12}}+\frac{AM^2}{bLN}+\frac{M^2}{L} + \frac{b^{\frac34}AM^{\frac12}N^{\frac54}}{L^{\frac12}}+b^{\frac12}AL^{\frac32}N^{\frac74}+\frac{b^{\frac12}A^{\frac12}MN}{L}\bigg). \tag{5.1}$$

We wish to choose L so that

$$b^{\frac12}AL^{\frac32}N^{\frac74}\approx\frac{AM^2}{bLN}+\frac{M^2}{L}+\frac{b^{\frac12}A^{\frac12}MN}{L},$$

and $L\geq 2\log(b\vartheta M)$. So we take

$$L=\frac{M^{\frac45}}{b^{\frac35}N^{\frac{11}{10}}}+\frac{M^{\frac45}}{b^{\frac15}A^{\frac25}N^{\frac{7}{10}}}+\frac{M^{\frac25}}{A^{\frac15}N^{\frac{3}{10}}}+M^{\varepsilon}.$$

With this choice (5.1) implies

$$\begin{aligned}\mathcal{C}_b &\ll \|\beta\|^2\|\nu\|^2M^{\varepsilon}\Big(1+\frac{|\vartheta|A}{bNM}\Big)^{\frac12}\bigg(AM(bN)^{\frac12}+b^{\frac34}AM^{\frac12}N^{\frac54} \\ &\quad + b^{\frac12}AN^{\frac74}\Big(\frac{M^{\frac45}}{b^{\frac35}N^{\frac{11}{10}}}+\frac{M^{\frac45}}{b^{\frac15}A^{\frac25}N^{\frac{7}{10}}}+\frac{M^{\frac25}}{A^{\frac15}N^{\frac{3}{10}}}+M^{\varepsilon}\Big)^{\frac32}\bigg) \\ &\ll \|\beta\|^2\|\nu\|^2M^{\varepsilon}\Big(1+\frac{|\vartheta|A}{bNM}\Big)^{\frac12}\bigg(AM(bN)^{\frac12}+b^{\frac34}AM^{\frac12}N^{\frac54} \\ &\quad + \frac{AM^{\frac65}N^{\frac{1}{10}}}{b^{\frac25}}+b^{\frac15}A^{\frac25}M^{\frac65}N^{\frac{7}{10}}+A^{\frac{7}{10}}b^{\frac12}M^{\frac35}N^{\frac{13}{10}}+b^{\frac12}AN^{\frac74}\bigg).\end{aligned} \tag{5.2}$$

6 Removing the square-free condition

We write $n = bn'$, where n' is square-free, b is square-full, and $(b, n') = 1$. We have

$$\Bigg|\sum_{a\in\mathcal{A},}\sum_{\substack{n\in\mathcal{N},\\(m,n)=1}}\beta_n\nu_a\,\mathrm{e}\bigg(\vartheta a\frac{\overline{m}}{n}\bigg)\Bigg|^2 = \Bigg|\sum_{\substack{b\le N,\\(b,\vartheta)=1,\\b\text{ square-full}}}\sum_{a\in\mathcal{A},}\sum_{\substack{bn'\in\mathcal{N},\\(b\vartheta m,n')=1,\\n'\text{square-free}}}\beta_{bn'}\nu_a\,\mathrm{e}\bigg(\vartheta a\frac{\overline{m}}{bn'}\bigg)\Bigg|^2$$

$$\ll\Bigg(\sum_{\substack{b\le N,\\b\text{ square-full}}}b^{-\frac{1}{2}}\Bigg)\Bigg(\sum_{\substack{b\le N,\ (b,\vartheta)=1\\b\text{ square-full}}}b^{\frac{1}{2}}\Bigg|\sum_{a\in\mathcal{A},}\sum_{\substack{bn'\in\mathcal{N},\\(b\vartheta m,n')=1}}\mu(n')^2\beta_{bn'}\nu_a\,\mathrm{e}\bigg(\vartheta a\frac{\overline{m}}{bn'}\bigg)\Bigg|^2\Bigg)$$

$$\ll\sum_{\substack{b\le N,\ (b,\vartheta)=1\\b\text{ square-full}}}b^{\frac{1}{2}}\Bigg|\sum_{a\in\mathcal{A},}\sum_{\substack{bn'\in\mathcal{N},\\(b\vartheta m,n')=1}}\mu(n')^2\beta_{bn'}\nu_a\,\mathrm{e}\bigg(\vartheta a\frac{\overline{m}}{bn'}\bigg)\Bigg|^2 M^{\varepsilon}.$$

Thus,

$$\begin{aligned}\mathcal{C}_1(M,N,A;\beta,\nu) &= \sum_{m\in\mathcal{M}}\Bigg|\sum_{a\in\mathcal{A},}\sum_{\substack{n\in\mathcal{N},\\(m,n)=1}}\beta_n\nu_a\,\mathrm{e}\bigg(a\frac{\overline{m}}{n}\bigg)\Bigg|^2\\ &\ll\sum_{\substack{b\le N,\ (b,\vartheta)=1,\\b\text{ square-full}}}b^{\frac{1}{2}}\sum_{\substack{m\in\mathcal{M},\\(m,b)=1}}\Bigg|\sum_{a\in\mathcal{A}}\sum_{\substack{n'\in\mathcal{N}_b,\\(b\vartheta m,n')=1}}\mu(n')^2\beta_{bn'}\nu_a\,\mathrm{e}\bigg(\vartheta a\frac{\overline{m}}{bn'}\bigg)\Bigg|^2 M^{\varepsilon}\\ &= M^{\varepsilon}\sum_{\substack{b\le N,\ (b,\vartheta)=1\\b\text{ square-full}}}b^{\frac{1}{2}}\mathcal{C}_b(M,N/b,A;\beta_b,\nu)\end{aligned}$$

where $\beta_b(n) := \mu(n)^2\beta_{bn}$ and $\mathcal{N}_b := [N/(2b), N/b]$. Now, if $b \le B$, we apply (5.2) and obtain that the contribution of those terms to $\mathcal{C}_1$ is bounded by

$$\begin{aligned}
&\ll \|\nu\|^2 M^\varepsilon \Big(1+\frac{|\vartheta|A}{NM}\Big)^{\frac12} \sum_{\substack{b\le B,\\ b\text{ square-full}}} b^{\frac12}\|\beta_b\|^2\Big(AMN^{\frac12}+\frac{AM^{\frac12}N^{\frac54}}{b^{\frac12}}\\
&\qquad\qquad + \frac{AM^{\frac65}N^{\frac1{10}}}{b^{\frac12}} + \frac{A^{\frac25}M^{\frac65}N^{\frac7{10}}}{b^{\frac12}} + \frac{A^{\frac7{10}}M^{\frac35}N^{\frac{13}{10}}}{b^{\frac45}} + \frac{AN^{\frac74}}{b^{\frac54}}\Big) \qquad (6.1)\\
&\ll \|\beta\|^2\|\nu\|^2 M^\varepsilon \Big(1+\frac{|\vartheta|A}{NM}\Big)^{\frac12}\Big(AMN^{\frac12}B^{\frac12}+AM^{\frac12}N^{\frac54}+AM^{\frac65}N^{\frac1{10}}+\\
&\qquad\qquad + A^{\frac25}M^{\frac65}N^{\frac7{10}} + A^{\frac7{10}}M^{\frac35}N^{\frac{13}{10}} + AN^{\frac74}\Big).
\end{aligned}$$

For $b > B$ we apply the trivial bound $\mathcal{C}_b(M, N/b, A, \beta_b, \nu) \ll \|\nu\|^2 A\frac{MN}{b}\|\beta_b\|$ and get that these terms contribute

$$\ll \|\nu\|^2 ANM^{1+\varepsilon} \sum_{\substack{b>B,\\ b\text{ square-full}}} \|\beta_b\|^2 \frac{1}{b^{\frac12}} \ll \|\beta\|^2\|\nu\|^2 \frac{ANM^{1+\varepsilon}}{B^{\frac12}}. \qquad (6.2)$$

We choose $B = N^{\frac12}$, so that combining (6.2) and (6.1) we obtain

$$\begin{aligned}
\mathcal{C}_1 &\ll \|\beta\|^2\|\nu\|^2 M^\varepsilon \Big(1+\frac{|\vartheta|A}{NM}\Big)^{\frac12}\\
&\quad \times \big(AMN^{\frac34} + AM^{\frac12}N^{\frac54} + AM^{\frac65}N^{\frac1{10}} + A^{\frac25}M^{\frac65}N^{\frac7{10}} + A^{\frac7{10}}M^{\frac35}N^{\frac{13}{10}} + AN^{\frac74}\big).
\end{aligned}$$

Notice that the third summand on the second line can be absorbed. Indeed, if $M \ll N^2$, then $AM^{\frac65}N^{\frac1{10}} \ll AMN^{\frac34}$, whereas if $M \gg N^2$ then one can obtain a stronger bound from Theorem 5 of [DFI97], which gives

$$\mathcal{C}_1 \ll \|\beta\|^2\|\nu\|^2 AM^{1+\varepsilon} \qquad (6.3)$$

in such range. Moreover, we also have $AM^{\frac12}N^{\frac54} \ll AMN^{\frac34} + AN^{\frac74}$ and thus

$$\begin{aligned}
\mathcal{C}_1 &\ll \|\beta\|^2\|\nu\|^2 (MN)^\varepsilon \Big(1+\frac{|\vartheta|A}{NM}\Big)^{\frac12}\\
&\quad \times \big(AMN^{\frac34} + AN^{\frac74} + A^{\frac25}M^{\frac65}N^{\frac7{10}} + A^{\frac7{10}}M^{\frac35}N^{\frac{13}{10}}\big),
\end{aligned} \qquad (6.4)$$

and this bound holds also without the assumption $\vartheta, A, N \ll M^C$, for some $C > 0$, since it is trivial otherwise.

7 Completion of the proof of Theorem 1

Combining the bound (6.4) for $\mathcal{C}_1$ and (2.1), we deduce

$$\begin{aligned}\mathcal{B}(M,N,A) \ll \|\alpha\|\|\beta\|\|\nu\|(AMN)^{\varepsilon}\Big(1+\frac{|\vartheta|A}{NM}\Big)^{\frac14} \\ \big(A^{\frac12}M^{\frac12}N^{\frac38}+A^{\frac12}N^{\frac78}+A^{\frac15}M^{\frac35}N^{\frac{7}{20}}+A^{\frac{7}{20}}M^{\frac{3}{10}}N^{\frac{13}{20}}\big).\end{aligned} \tag{7.1}$$

If $M \geq N$ this implies

$$\mathcal{B}(M,N,A) \ll \|\alpha\|\|\beta\|\|\nu\|(MN)^{\varepsilon}\Big(1+\frac{|\vartheta|A}{MN}\Big)^{\frac14}\big(A^{\frac12}M^{\frac12}N^{\frac38}+A^{\frac{7}{20}}M^{\frac35}N^{\frac{7}{20}}\big),$$

which is slightly stronger than the bound stated in Theorem 1 in the range $M \geq N$. We then use the following remark to obtain (1.2) also in the range $M < N$.

Remark 2. *All the computations of Section 3 and 4 work, applying partial summation at appropriate places, also when an extra addend $f_{a,\vartheta}(m,bn)$ (with $f_{a,\vartheta}(x,y)$ satisfying (1.3)) is inserted in the exponential function in the definition (2.2) of $\mathcal{C}_b$. Thus, one arrives at the bound*

$$\begin{aligned}\mathcal{B}(f;M,N,A) \ll \|\alpha\|\|\beta\|\|\nu\|(MN)^{\varepsilon}\Big(1+\frac{|\vartheta|A+X}{MN}\Big)^{\frac12} \\ \times\big(A^{\frac12}M^{\frac12}N^{\frac38}+A^{\frac{7}{20}}M^{\frac35}N^{\frac{7}{20}}\big)\end{aligned} \tag{7.2}$$

in the case $M \geq N$. Notice that the exponent of $(1+\frac{|\vartheta|A+X}{MN})$ is now $\frac12$ because in this case we need to apply partial summation also when dealing with the diagonal term and for the analogue of (6.3). For example, in Section 3, when we add $f_{a,\vartheta}(m,bn)$, the sum over m in (3.2) becomes

$$\Bigg|\sum_{\substack{m\in\mathcal{M},\\ (m,b\ell_1\ell_2n_1n_2)=1}} \mathrm{e}\Big(\vartheta\frac{(a_1\ell_1-a_2\ell_2)\overline{m}}{b\ell_1n_1}\Big)\,\mathrm{e}\big(f_{a_1,\vartheta}(m,bn_1)-f_{a_2,\vartheta}(m,bn_2)\big)\Bigg|.$$

From the condition of $f_{a,\vartheta}$, we have that $\frac{\partial}{\partial x}f_{a,\vartheta}(x,bn) \ll \frac{X}{x^2bn}$. Applying Weil's bound in Lemma 3 and partial summation to the terms satisfying $a_1\ell_1 \neq a_2\ell_2$, we obtain that the sum over m above is bounded by

$$\Big((bLN)^{\frac12+\varepsilon}+(a_1\ell_1-a_2\ell_2,bn_1\ell_1)\frac{M^{1+\varepsilon}}{bLN}\Big)\Big(1+\frac{X}{bMN}\Big).$$

We now observe that the elementary reciprocity law allows us to write $\mathcal{B}(M,N,A)$ as

$$\mathcal{B}(M,N,A) = \sum\sum\sum_{\substack{m\in\mathcal{M},n\in\mathcal{N},a\in\mathcal{A},\\(m,n)=1}} \alpha_m\beta_n\nu_a \,\mathrm{e}\Big(-\vartheta\frac{a\overline{n}}{m} + \frac{\vartheta a}{mn}\Big).$$

Thus, if $M < N$ we can apply (7.2) with the role of M and N switched and with $f_{a,\vartheta}(x,y) := \frac{\vartheta a}{xy}$. Since $f_{a,\vartheta}(x,y)$ satisfies the condition (1.3) with $X = |\vartheta|A$, we obtain (1.2) for $M < N$, and so the proof of Theorem 1 is complete.

8 Proof of Theorem 2

We proceed in the same way as in the proof of Theorem 1, with very minor modifications. First, we assume $bN \geq M$ and in the off-diagonal term we split the sum modulo m into classes $m \equiv c \pmod{4b\mathfrak{q}_1\mathfrak{p}_2}$ with $(m, b\mathfrak{q}_1\mathfrak{p}_2) = 1$ rather than modulo $b\mathfrak{q}_1\mathfrak{p}_2$. Notice that this implies $\tilde{\ell}_2\mathfrak{q}_2n_2' \equiv \tilde{\ell}_1\mathfrak{p}_1n_1' - cd \pmod{4d}$. Then, we can repeat the same arguments keeping the Jacobi symbol $(\frac{m}{n_1n_2})$ inside the summations, so that in the analogue of Section 4.1.1 this factor will become

$$\begin{aligned}\Big(\frac{m}{n_1n_2}\Big) &= \Big(\frac{c}{\mathfrak{q}_1\mathfrak{p}_2}\Big)\Big(\frac{m}{\mathfrak{p}_1\mathfrak{q}_2n_1'n_2'}\Big) = \Big(\frac{c}{\mathfrak{q}_1\mathfrak{p}_2}\Big)\Big(\frac{d}{\mathfrak{p}_1\mathfrak{q}_2n_1'n_2'}\Big)\Big(\frac{dm}{\mathfrak{p}_1\mathfrak{q}_2n_1'n_2'}\Big)\\ &= \Big(\frac{c}{\mathfrak{q}_1\mathfrak{p}_2}\Big)\Big(\frac{d}{\mathfrak{p}_1\mathfrak{q}_2n_1'}\Big)\Big(\frac{d}{\tilde{\ell}_2\mathfrak{q}_2n_2'}\Big)\Big(\frac{d}{\tilde{\ell}_2\mathfrak{q}_2}\Big)\Big(\frac{-\tilde{\ell}_2\mathfrak{q}_2n_2'}{\mathfrak{p}_1n_1'}\Big)\Big(\frac{\tilde{\ell}_1\mathfrak{p}_1n_1'}{\mathfrak{q}_2n_2'}\Big)\\ &= f(c,\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2,n_1',n_2')g(c,\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2,n_1',\ell_1,\ell_2)\Big(\frac{\tilde{\ell}_1}{n_2'}\Big),\end{aligned}$$

where

$$\begin{aligned}f(\dots) &:= \Big(\frac{c}{\mathfrak{q}_1\mathfrak{p}_2}\Big)\Big(\frac{-\mathfrak{q}_2n_2'}{\mathfrak{p}_1n_1'}\Big)\Big(\frac{\mathfrak{p}_1n_1'}{\mathfrak{q}_2n_2'}\Big),\\ g(\dots) &:= \Big(\frac{d}{\mathfrak{p}_1\mathfrak{q}_2n_1'}\Big)\Big(\frac{d}{\tilde{\ell}_1\mathfrak{p}_1n_1' - cd}\Big)\Big(\frac{d}{\tilde{\ell}_2\mathfrak{q}_2}\Big)\Big(\frac{\tilde{\ell}_2}{\mathfrak{p}_1n_1'}\Big)\Big(\frac{\tilde{\ell}_1}{\mathfrak{q}_2}\Big),\end{aligned}$$

and where we used that $\left(\frac{a}{b}\right)$ is periodic modulo b in a if $b \geq 1$ and is periodic modulo $4|a|$ in b if $a \neq 0$.

Thus, we arrive at the analogue of (4.25) which will include an extra factor of $(\frac{\tilde{\ell}_1}{n_2'})$. The rest of the argument is identical, with the difference that we apply (A.2) rather than (A.1). The fact that the former bound is weaker than the latter does not affect the arguments when $bN \geq M$. Thus, we obtain (7.1) and whence (1.5) when $N \geq M$. This also implies (1.5) for $M > N$, as can be seen by using the reciprocity formulae for the Jacobi symbol and for $\frac{\overline{m}}{n}$ and applying the bound for the case $N \geq M$.

9 Proof of Corollary 1

Proceeding as in [DFI95], we see that the error term in (1.4) is equal to

$$\sum_{d|\Delta}\int_0^{4D}\sum\sum\sum_{\substack{dn_1\in\mathcal{N}_1, dn_2\in\mathcal{N}_2, h\in\mathbb{Z}\\ 1\leq|h|\leq HD^{\varepsilon}, (n_1,n_2)=1}}\frac{\tilde{\alpha}_{dn_1}(x)\tilde{\beta}_{dn_2}(x+\Delta)}{dn_1n_2}\,\mathrm{e}\bigg(\frac{h\Delta}{d}\frac{\overline{n}_1}{n_2}\bigg)\mathrm{e}\bigg(\frac{hx}{dn_1n_2}\bigg)\,dx+O(\eta),$$

with $H=\frac{\eta}{d}(\frac{N_1}{M_1}+\frac{N_2}{M_2})$, $D=M_1N_2+M_2N_1$ and $\alpha_r(x):=\alpha_r g(x/r)$, $\tilde{\beta}_r:=\beta_r f(x/r)$. Applying Theorem 1 (in the version given by Remark 1) we see that we can bound the above sum by

$$\ll\sum_{d|\Delta}\sum_{1\leq|h|\leq HD^{\varepsilon}}\frac{dD}{N_1N_2}\frac{\|\alpha\|\|\beta\|}{d^{\frac{1}{2}}}\bigg(1+\frac{dHD}{N_1N_2}\bigg)^{\frac{1}{2}}(N_1N_2)^{\frac{7}{20}}(N_1+N_2)^{\frac{1}{4}+\varepsilon},$$

$$\ll\eta^{\frac{3}{2}}\bigg(\frac{M_1N_2}{M_2N_1}+\frac{M_2N_1}{M_1N_2}\bigg)^{\frac{3}{2}}\|\alpha\|\|\beta\|(N_1N_2)^{\frac{7}{20}}(N_1+N_2)^{\frac{1}{4}+\varepsilon}(M_1M_2)^{\varepsilon},$$

since we can assume $|\Delta|\leq 4D$ (otherwise both $\mathcal{T}$ and the main term on the right hand side of (1.4) are identically zero) and since

$$\frac{dHD}{N_1N_2}=\eta\bigg(\frac{N_1}{M_1}+\frac{N_2}{M_2}\bigg)\bigg(\frac{M_1}{N_1}+\frac{M_2}{N_2}\bigg)\leq 2\eta\bigg(\frac{M_1N_2}{M_2N_1}+\frac{M_2N_1}{M_1N_2}\bigg).$$

A Appendix: Weil's bound for incomplete Kloosterman sums

In this appendix we give the following bounds for incomplete Kloosterman sums where the sums are subject to some coprimality and congruence conditions.

Lemma 3. *Let $\delta, k, \gamma \geq 1$, $\alpha, a, b, v \in \mathbb{Z}$ and let $I = \{x \in [X', X'+X], x \equiv v \pmod k\}$. Let $h := (k,\gamma)$, $h_1 := (k^\infty, \gamma)$, $\gamma_1 := \frac{\gamma}{h_1}$, where $(m^\infty, n) := \lim_{r\to\infty}(m^r, n)$. Then,*

$$\sum_{\substack{x\in I,\\ (x,\gamma\delta)=1}} \mathrm{e}\left(\frac{\alpha\overline{x}}{\gamma}\right) \ll (\gamma\delta)^\varepsilon \frac{h_1}{h}\left(\frac{\gamma_1}{(\alpha,\gamma_1)}\right)^{\frac{1}{2}} + (\alpha,\gamma_1)\frac{X\delta^\varepsilon}{\gamma_1 k}. \tag{A.1}$$

Moreover, if χ is a Dirichlet character modulo γ and $c, d \geq 1$, $a, b, \beta \in \mathbb{Z}$, then

$$\sum_{\substack{x\in I,\\ (x,\gamma\delta)=1,\\ (ax+b,c)=d}} \chi(x)\,\mathrm{e}\left(\frac{\alpha\overline{x}+\beta x}{\gamma}\right) \ll (c\gamma\delta)^\varepsilon \frac{h_1}{h}\left(\frac{\gamma_1}{(\alpha,\gamma_1)}\right)^{\frac{1}{2}} + (\alpha,\gamma_1)^{\frac{1}{2}}\gamma_1^{\frac{1}{2}+\varepsilon}\frac{X(c\delta)^\varepsilon}{\gamma_1 k}. \tag{A.2}$$

Proof. We start by proving (A.1). First of all, we notice that we can assume that $(\delta, \gamma k) = (v, h) = 1$ and $k \leq X$. Also, the case $\delta > 1$ can be easily obtained from the case $\delta = 1$ by Möbius inversion. Indeed

$$\begin{aligned}\sum_{\substack{x\in I,\\ (x,\gamma\delta)=1}} \mathrm{e}\left(\frac{\alpha\overline{x}}{\gamma}\right) &= \sum_{r_1|\delta}\mu(r_1)\sum_{\substack{x\in I,\ (x,\gamma)=1,\\ x\equiv 0 \pmod{r_1}}} \mathrm{e}\left(\frac{\alpha\overline{x}}{\gamma}\right)\\ &\ll \sum_{d|\delta}\left(\gamma^\varepsilon\frac{h_1}{h}\left(\frac{\gamma_1}{(\alpha,\gamma_1)}\right)^{\frac{1}{2}+\varepsilon} + (\alpha,\gamma_1)\frac{X}{\gamma_1 k}\right)\\ &\ll (\gamma\delta)^\varepsilon\frac{h_1}{h}\left(\frac{\gamma_1}{(\alpha,\gamma_1)}\right)^{\frac{1}{2}} + \frac{(\alpha,\gamma_1)X\delta^\varepsilon}{\gamma_1 k},\end{aligned}$$

by (A.1) in the case $\delta = 1$. Similarly, the case $(k,\gamma) > 1$ can be recovered from the case $h = (k,\gamma) = 1$:

$$\begin{aligned}\sum_{\substack{x\in I,\\ (x,\gamma)=1}} \mathrm{e}\left(\frac{\alpha\overline{x}}{\gamma}\right) &= \sideset{}{^*}\sum_{\substack{c \pmod{h_1},\\ c\equiv v \pmod h}}\ \sum_{\substack{x\in I_c,\\ (x,\gamma_1)=1}} \mathrm{e}\left(\frac{\alpha\overline{x}}{\gamma}\right) = \sideset{}{^*}\sum_{\substack{c \pmod{h_1},\\ c\equiv v \pmod h}} \mathrm{e}\left(\frac{\alpha\overline{c\gamma_1}}{h_1}\right)\sum_{\substack{x\in I_c,\\ (x,\gamma_1)=1}} \mathrm{e}\left(\frac{\alpha\overline{h_1 x}}{\gamma_1}\right)\\ &\ll \sideset{}{^*}\sum_{\substack{c \pmod{h_1},\\ c\equiv v \pmod h}}\left(\frac{\gamma_1^{\frac{1}{2}+\varepsilon}}{(\alpha,\gamma_1)^{\frac{1}{2}}} + \frac{(\alpha,\gamma_1)X}{\gamma_1 k\frac{h_1}{h}}\right) \ll \delta^\varepsilon\frac{h_1}{h}\frac{\gamma_1^{\frac{1}{2}+\varepsilon}}{(\alpha,\gamma_1)^{\frac{1}{2}}} + \delta^\varepsilon(\alpha,\gamma_1)\frac{X}{\gamma_1 k},\end{aligned}$$

where $I_c := I \cap \{x \equiv c \pmod{h_1}\}$. Thus, we can assume $h = 1$ and similarly also that $(\alpha, \gamma) = 1$. Now, applying the Erdös-Turán inequality as in Lemma 8 of [DFI97], we find

$$\Big| \sum_{\substack{x \in I, \\ (x,\gamma)=1}} \mathrm{e}\Big(\frac{\alpha \overline{x}}{\gamma}\Big) \Big| \leq \frac{X+k}{\gamma k} |S(\alpha, 0; \gamma)| + \sum_{1 \leq r \leq \frac{\gamma}{2}} \frac{1}{r} |S(\alpha, r\overline{k}; \gamma)|.$$

Thus, using Weil's bound for $|S(\alpha, r\overline{k}; \gamma)|$ and observing that $|S(\alpha, 0; \gamma)|$ is a Ramanujan sum and thus is bounded by (α, γ), we obtain (A.1).

To prove (A.2) we can proceed in a similar way. As before, we assume $(\delta, \gamma k) = (v, h) = 1$, $k \leq X$ and $\delta = 1$. Also, we can assume $d|c$, $(a, c) = 1$.

Moreover, we can deduce the case $c > 1$ from the case $c = 1$ as the following. For the case $\delta > 1$, we start by using Möbius inversion and we obtain

$$\sum_{\substack{x \in I, \\ (x,\gamma)=1, \\ (ax+b,c)=d}} \chi(x)\, \mathrm{e}\Big(\frac{\alpha \overline{x} + \beta x}{\gamma}\Big) = \sum_{\substack{r|c, \\ d|r}} \mu(r) \sum_{\substack{x \in I,\ (x,\gamma)=1, \\ x \equiv -\overline{a}b \pmod{r}}} \chi(x)\, \mathrm{e}\Big(\frac{\alpha \overline{x} + \beta x}{\gamma}\Big). \tag{A.3}$$

Next, we let $r = g_1 g_2$, where g_1 is the largest divisor of r such that $(\gamma/(\gamma, g_1), g_1) = 1$. Also, we write $\gamma = f_1 f_2$ where $f_1 = (\gamma, g_1)$ and notice that $(g_1, g_2) = (f_1, f_2) = 1$ and $f_1 | g_1$, $g_2 | f_2$ (and thus also $(f_1, g_2) = (g_1, f_2) = 1$). Also, we can write χ as $\chi_1 \chi_2$, where for $i = 1, 2$, χ_i is a Dirichlet character modulo f_i. Using the orthogonality of additive character we obtain that the left hand side of (A.3) is equal to

$$\begin{aligned} &\sum_{\substack{r|c, \\ d|r}} \frac{\mu(r)}{g_2} \sum_{u \pmod{g_2}} \mathrm{e}\Big(\frac{u \overline{a} b}{g_2}\Big) \sum_{\substack{x \in I,\ (x,\gamma)=1, \\ x \equiv -\overline{a}b \pmod{g_1}}} \chi(x)\, \mathrm{e}\Big(\frac{\alpha \overline{x} + \beta x}{\gamma} + \frac{ux}{g_2}\Big) \\ &\leq \sum_{\substack{r|c, \\ d|r}} \frac{1}{g_2} \sum_{u \pmod{g_2}} \Big| \sum_{\substack{x \in I,\ (x,\gamma)=1, \\ x \equiv -\overline{a}b \pmod{g_1}}} \chi_2(x)\, \mathrm{e}\Big(\frac{\alpha \overline{f_1 x} + (\beta \overline{f_1} + u f_2/g_2) x}{f_2}\Big) \Big|, \end{aligned}$$

since $x \equiv -\overline{a}b \pmod{f_1}$ and thus the Chinese remainder theorem gives

$$\chi(x)\, \mathrm{e}\Big(\frac{\alpha \overline{x} + \beta x}{\gamma}\Big) = \chi_1(-\overline{a}b)\, \mathrm{e}\Big(-\frac{\alpha a \overline{b} \overline{f_2} + \beta a \overline{b} f_2}{f_1}\Big) \chi_2(x)\, \mathrm{e}\Big(\frac{\alpha \overline{f_1 x} + \beta \overline{f_1} x}{f_2}\Big).$$

Thus, applying (A.2) in the case $c=1$ we obtain

$$\sum_{\substack{x\in I,\\(x,\gamma)=1,\\(ax+b,c)=d}} \chi(x)\,\mathrm{e}\bigg(\frac{\alpha\overline{x}+\beta x}{\gamma}\bigg) \ll \sum_{r|c}\bigg(\gamma^{\varepsilon}\frac{h_1'}{h'}\bigg(\frac{\gamma_1'}{(\alpha,\gamma_1')}\bigg)^{\frac12} + (\alpha,\gamma_1')^{\frac12}{\gamma_1'}^{\frac12+\varepsilon}\frac{X}{\gamma_1'k'}\bigg)$$
$$\ll (\gamma c)^{\varepsilon}\frac{h_1}{h}\bigg(\frac{\gamma_1}{(\alpha,\gamma_1)}\bigg)^{\frac12} + (\alpha,\gamma_1)^{\frac12}{\gamma_1}^{\frac12+\varepsilon}\frac{X}{\gamma_1 k},$$

where we used the notation

$$k' := [g_1,k], \quad h' = (k',f_2) := (k,f_2), \quad h_1' := ({k'}^{\infty},f_2) = (k^{\infty},f_2), \quad \gamma_1' := f_2/h_1'$$

since $(c,k)=1$. Moreover $\gamma_1'|\gamma_1$ and ${\gamma_1'}^{\frac12}k' = [g_1,k]f_2^{\frac12}(k^{\infty},f_2)^{-\frac12} \geq \gamma_1^{\frac12}k$ since $f_1|g_1$ and thus $[g_1,k]\geq[f_1,k] = f_1k/(k,f_1) \geq f_1^{\frac12}k(k,f_1)^{-\frac12}$.

The rest of the argument is essentially identical as in the case of (A.1), with the only difference being in the last step, where we need to use Weil's bound for both summands. □

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