AN OPTIMAL CHOICE OF DIRICHLET POLYNOMIALS FOR THE NYMAN-BEURLING CRITERION

S. BETTIN, J. B. CONREY, D. W. FARMER

In memory of Professor A. A. Karatsuba on the 75th anniversary of his birth

ABSTRACT. We give a conditional result on the constant in the Báez-Duarte reformulation of the Nyman-Beurling criterion for the Riemann Hypothesis. We show that assuming the Riemann hypothesis and that $\sum_\rho \frac{1}{|\zeta'(\rho)|^2} \ll T^{3/2-\delta}$, for some $\delta > 0$, the value of this constant coincides with the lower bound given by Burnol.

1. INTRODUCTION

The Nyman-Beurling-Báez-Duarte approach to the Riemann hypothesis asserts that the Riemann hypothesis is true if and only if

$$\lim_{N\to\infty} d_N^2 = 0,$$

where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi}\int_{-\infty}^{\infty} |1-\zeta A_N(1/2+it)|^2 \frac{dt}{\frac{1}{4}+t^2}$$

and the infimum is over all Dirichlet polynomials $A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ of length N (see [Bag] for a nice account of this).

An open question is to determine what the rate of convergence of d_n to zero is, assuming the Riemann hypothesis. Balazard and de Roton showed that, if the Riemann hypothesis is true, then

$$d_N^2 \ll \frac{(\log\log N)^{\frac{5}{2}+\varepsilon}}{\sqrt{\log N}},$$

for all $\varepsilon > 0$. On the other hand Báez-Duarte, Balazard, Landreau and Saias [BBLS00, BBLS05] showed (unconditionally) that d_N^2 can not decay faster than a constant times $\frac{1}{\log N}$. More precisely, they showed that

$$\liminf_{N\to\infty} d_N^2 \log N \geq \sum_{\Re(\rho)=1/2} \frac{1}{|\rho|^2},$$

where here and in the following the sum is restricted to distinct zeros of the Riemann zeta function on the critical line. The constant was later improved by Burnol [Bur] who showed

$$\liminf_{N\to\infty} d_N^2 \log N \geq \sum_{\Re(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2},$$

where $m(\rho)$ denotes the multiplicity of ρ. This lower bound is believed to be optimal and one expects that

$$d_N^2 \sim \frac{1}{\log N} \sum_{\Re(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2}. \tag{1}$$

Notice that under the Riemann hypothesis, one has

$$\sum_{\Re(\rho)=1/2} \frac{m(\rho)}{|\rho|^2} = 2 + \gamma - \log 4\pi$$

and in particular, if all the non-trivial zeros of $\zeta(s)$ are simple, then (1) can be rewritten as

$$d_N^2 \sim \frac{2+\gamma-\log 4\pi}{\log N}.$$

It is the purpose of this note to prove (1) under the Riemann Hypothesis and assuming a mild condition on the growth of the mean value of $\frac{1}{|\zeta'(\rho)|^2}$ over the non-trivial zeros $|\rho| \leq T$ of $\zeta(s)$. This will be achieved by using the Dirichlet polynomial

$$V_N(s) := \sum_{n=1}^{N} \left(1 - \frac{\log n}{\log N}\right) \frac{\mu(n)}{n^s}.$$

Theorem 1. *If the Riemann hypothesis is true and if*

$$\sum_{|\Im(\rho)|\leq T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{\frac{3}{2}-\delta} \tag{2}$$

for some $\delta > 0$, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |1 - \zeta V_N(1/2+it)|^2 \frac{dt}{\frac{1}{4}+t^2} \sim \frac{2+\gamma-\log 4\pi}{\log N}.$$

The condition (2) implicitly assumes that the zeros of the Riemann zeta function are all simple. Moreover, this upper bound is "mild" in the sense that a conjecture, due to Gonek and recovered by a different heuristic method of Hughes, Keating, and O'Connell [HKO], predicts that

$$\sum_{|\rho|\leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{6}{\pi^3} T.$$

We remark that Theorem 1 is in contrast to what one might have expected after viewing the graphs of Landreau and Richards [LR] which at first sight suggest that V_N is not optimal.

This behaviour of the Riemann zeta function resembles that of polynomials. In fact, Grenander and Rosenblatt [GR] (see also Theorem 2.1 in [Bur]) showed that for a polynomial $P(z)$ one has that the zeros of P are all located outside or on the unit circle if and only if $\lim_{N\to\infty}\delta_N = 0$, where

$$\delta_N^2 = \frac{1}{2\pi}\inf_{Q_N}\int_0^{2\pi}|1-P(z)\,Q_N(z)|^2\,\mathrm{d}\theta,$$

where $z=e^{i\theta}$ and the infimum is over polynomials Q_N of degree at most N. Moreover, if this happens, then

$$\lim_{N\to\infty} N\delta_N^2 = \sum_{|\rho|=1} m(\rho)^2,$$

where the sum is restricted to the distinct zeros ρ of $P(z)$ lying on the unit circle and $m(\rho)$ is again the multiplicity of ρ.

This analogy seems to apply also to the choices of optimal polynomials.

Theorem 2. *Let $P(z)$ be a polynomial whose zeros are all simple and lie outside or on the unit circle. Let*

$$W_N(z) := \sum_{n=0}^{N}\Big(1-\frac{n}{N}\Big)a_n z^n, \tag{3}$$

where

$$\frac{1}{P(z)} = \sum_{n\geq 0} a_n z^n$$

is the Taylor expansion in $x=0$ of the inverse of $P(z)$ (i.e. it is the formal power series inverse of $P(z)$). Then

$$\frac{1}{2\pi}\int_0^{2\pi}|1-P(z)W_N(z)|^2\,\mathrm{d}\theta \sim \frac{1}{N}\sum_{|\rho|=1} m(\rho)^2,$$

where $z=e^{i\theta}$.

We remark that the proofs of Theorem 1 and 2 are very similar, the main difference being that the Riemann zeta function has infinitely many zeros. This generates some issues concerning the convergence of certain sums of $\frac{1}{\zeta'(\rho)}$, which force us to assume condition (2).

2. POLYNOMIALS

Lemma 1. *Let $P(s)$ be a polynomial with $P(0)\neq 0$. We have*

$$W_N(s) = \frac{1}{P(s)}\Big(1+\frac{s}{N}\frac{P'}{P}(s)\Big) - \frac{s}{N}Y_N(s),$$

where $W_N(s)$ *is defined in* (3),

$$Y_N(s) := \sum_{\rho} \operatorname*{Res}_{z=\rho} \frac{s^N}{P(z)(z-s)^2 z^N}$$

and the sum is over distinct zeros ρ *of* $P(z)$*.*

Proof. Since $P(0) \neq 0$, we can take an $\varepsilon > 0$ such that all the zeros of $P(z)$ lie outside of the circle $|z| = \varepsilon$. Now, observe that we can assume $0 < |s| < \varepsilon$, since the result will then extend to all $\mathbb{C}$ by analytic continuation. Denoting by $\mathcal{C}_y$ the circle of radius $y > 0$ (oriented in the positive direction), by the residue theorem we have that

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \frac{\mathrm{d}z}{z^{n+1}},$$

therefore

$$W_N(s) = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \sum_{n=0}^{N} \Big(1 - \frac{n}{N}\Big) \Big(\frac{s}{z}\Big)^n \frac{\mathrm{d}z}{z}.$$

Now,

$$\sum_{n=0}^{N} \Big(1 - \frac{n}{N}\Big) z^n = -\frac{1}{N} \frac{z - z^{N+1}}{(1-z)^2} + \frac{1}{1-z}$$

and thus

$$W_N(s) = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \bigg(-\frac{1}{N} \frac{s z^N - s^{N+1}}{(z-s)^2 z^N} + \frac{1}{z-s} \bigg) \mathrm{d}z.$$

Now, by the residue theorem

$$\frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \bigg(-\frac{1}{N} \frac{s}{(z-s)^2} + \frac{1}{z-s} \bigg) \mathrm{d}z = \frac{1}{P(s)} \bigg(1 + \frac{s}{N} \frac{P'(s)}{P(s)} \bigg),$$

whereas, moving the line of integration to $\mathcal{C}_y$ and letting y tend to infinity, one has that

$$\frac{1}{2\pi i N} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \frac{s^{N+1}}{(z-s)^2 z^N} \mathrm{d}z = -\frac{s}{N} Y_N(s)$$

and the Lemma follows. □

Proof of Theorem 2. Let $\delta > 1$ be such that $P(s)$ does not have any zero on $1 < |s| \leq \delta$. We have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |1 - P(z) W_N(z)|^2 \, \mathrm{d}\theta &= \frac{1}{2\pi i} \int_{\mathcal{C}_1} (1 - P(s) W_N(s)) \bigg(1 - \overline{P}\Big(\frac{1}{s}\Big) \overline{W}_N\Big(\frac{1}{s}\Big) \bigg) \frac{\mathrm{d}s}{s} \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}_\delta} (1 - P(s) W_N(s)) \bigg(1 - \overline{P}\Big(\frac{1}{s}\Big) \overline{W}_N\Big(\frac{1}{s}\Big) \bigg) \frac{\mathrm{d}s}{s}. \end{aligned}$$

Therefore, by Lemma 1, this is

$$\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\left(\frac{P'}{P}(s)-P(s)Y_N(s)\right)\left(\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right)-\overline{P}\left(\frac{1}{s}\right)\overline{Y}_N\left(\frac{1}{s}\right)\right)\frac{\mathrm{d}s}{s}.$$

Now, for $|s|=\delta$ one has

$$Y_N(s)\overline{Y}_N\left(\frac{1}{s}\right)=O(1),$$

therefore

$$\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\left(\frac{P'}{P}(s)\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right)+P(s)Y_N(s)\overline{P}\left(\frac{1}{s}\right)\overline{Y}_N\left(\frac{1}{s}\right)\right)\frac{\mathrm{d}s}{s}=O\left(\frac{1}{N^2}\right).$$

Moreover for $s\in\mathcal{C}_\delta$ one has that $\overline{Y}_N\left(\frac{1}{s}\right)=O\big(\delta^{-N}\big)$, thus

$$-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\left(\frac{P'}{P}(s)\overline{P}\left(\frac{1}{s}\right)\overline{Y}_N\left(\frac{1}{s}\right)\right)\frac{\mathrm{d}s}{s}=O\big(\delta^{-N}/N^2\big).$$

Finally, by the residue theorem,

$$\begin{aligned}
-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}&P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{s}=\\
&=-\frac{1}{N^2}\sum_{|\rho|=1}\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right)\frac{1}{s}+\\
&\quad-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_{\frac{1}{\delta}}}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{s}\\
&=-\frac{1}{N^2}\sum_{|\rho|=1}\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right)+O\big(\delta^{-N}/N^2\big).
\end{aligned}$$

The theorem then follows by observing that

$$\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right)\frac{1}{s}=-N+O(1).$$

□

3. The Riemann zeta-function

We start with the following lemma, which is the analogue of Lemma 1. We remark that this lemma is unconditional.

Lemma 2. *If $0 < \Re(s) < 1$, then*

$$V_N(s) = \frac{1}{\zeta(s)}\left(1 - \frac{1}{\log N}\frac{\zeta'}{\zeta}(s)\right) + \frac{1}{\log N}\sum_{\rho} R_N(\rho, s) + \frac{1}{\log N} F_s(1/N),$$

where the sum is over distinct non-trivial zeros ρ of $\zeta(s)$ with

$$R_N(\rho, s) = \operatorname*{Res}_{z=\rho} \frac{N^{z-s}}{\zeta(z)(z-s)^2},$$

and where

$$F_s(z) = \pi z^s \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n+1} z^{2n}}{(2n)!\zeta(2n+1)(2n+s)^2}$$

is an entire function of z.

Proof. We have

$$V_N(s) = \frac{1}{\log N}\frac{1}{2\pi i}\int_{(2)} \frac{N^w}{\zeta(s+w)}\frac{dw}{w^2},$$

where we use the notation $\int_{(c)}$ to mean an integration up the vertical line from $c - i\infty$ to $c + i\infty$. Now we move the path of integration to $\Re(w) = -\Re(s) - 2M - 1$ for a large integer M. The residue at $w = \rho - s$ is $R_N(\rho, s)/\log N$. The residue at $s + w = -2n$ is

$$\frac{N^{-2n-s}}{\zeta'(-2n)(2n+s)^2 \log N}$$

and the integral on the new path is $\ll N^{-2M-1}$. Letting $M \to \infty$ and using

$$\zeta'(-2n) = \frac{(-1)^n \pi (2n)! \zeta(2n+1)}{(2\pi)^{2n+1}}$$

we obtain the result. □

Lemma 3. *Let $\varepsilon > 0$. Assume the Riemann hypothesis and that all the zeros of $\zeta(s)$ are simple. Then, if condition* (2) *holds, for $\Re(s) = \frac{1}{2} \pm \varepsilon$ one has*

$$\sum_{\rho} R_N(\rho, s) \ll N^{\mp\varepsilon} |s|^{\frac{3}{4} - \frac{\delta}{2} + \varepsilon}. \tag{4}$$

Proof. Firstly observe that, by the Cauchy-Schwartz inequality, (2) implies

$$\sum_{|\rho| \le T} \frac{1}{|\zeta'(\rho)|} \ll \sqrt{N(T) \sum_{|\rho| \le T} \frac{1}{|\zeta'(\rho)|^2}} \ll T^{\frac{5}{4} - \frac{\delta}{2}} \sqrt{\log T},$$

since

$$N(T) := \frac{1}{2} \sum_{|\rho| \le T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Therefore, by partial summation, we have that the series

$$\sum_\rho \frac{1}{|\zeta'(\rho)||\rho|^\alpha}$$

is convergent for any $\alpha > \frac{5}{4} - \frac{\delta}{2}$. Now, for a simple zero ρ, we have

$$R_N(\rho, s) = \sum_\rho \frac{N^{\rho - s}}{\zeta'(\rho)(\rho - s)^2}.$$

Therefore

$$\begin{aligned} N^{\pm\varepsilon} \sum_\rho R_N(\rho, s) &\ll \sum_{|\rho - s| < \frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho - s|^2} + \sum_{|\rho - s| \geq \frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho - s|^2} \\ &\ll \sum_{|\rho - s| < \frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho - s|^2} + \sum_{|\rho - s| \geq \frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho|^2} \qquad (5) \\ &\ll \sum_{|\rho - s| < \frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho - s|^2} + 1. \end{aligned}$$

Now, by the Cauchy-Schwartz inequality,

$$\sum_{|\rho - s| < \frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho - s|^2} \ll \sqrt{\left(\sum_{|\rho| < 2|s|} \frac{1}{|\zeta'(\rho)|^2} \right) \left(\sum_{|\rho| < 2|s|} \frac{1}{|\rho - s|^4} \right)} \ll |s|^{\frac{3}{4} - \frac{\delta}{2} + \varepsilon},$$

since, by partial summation,

$$\sum_{|\rho| < 2|s|} \frac{1}{|\rho - s|^4} \ll \log(|s| + 2).$$

This completes the proof of the lemma. □

Proof of Theorem 1. We have

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} |1 - \zeta V_N(1/2 + it)|^2 \frac{dt}{1/4 + t^2} \\ &\qquad = \frac{1}{2\pi i} \int_{\left(\frac{1}{2}\right)} (1 - \zeta V_N(s))(1 - \zeta V_N(1 - s)) \frac{ds}{s(1 - s)} \\ &\qquad = \frac{1}{2\pi i} \int_{\left(\frac{1}{2} - \varepsilon\right)} (1 - \zeta V_N(s))(1 - \zeta V_N(1 - s)) \frac{ds}{s(1 - s)}. \end{aligned}$$

By Lemma 2, this is

$$\begin{aligned}\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{\left(\frac12-\varepsilon\right)}\left(\frac{\zeta'}{\zeta^2}(s)-\sum_\rho R_N(\rho,s)-F_s\left(\frac1N\right)\right)\times\\ \times\left(\frac{\zeta'}{\zeta^2}(1-s)-\sum_\rho R_N(\rho,1-s)-F_{1-s}\left(\frac1N\right)\right)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds.\end{aligned}\tag{6}$$

Now, we have

$$\begin{aligned}\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{\left(\frac12-\varepsilon\right)}\sum_{\rho_1,\rho_2}R_N(\rho_1,s)R_N(\rho_2,1-s)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds\\ \ll\frac{1}{\log^2 N}\int_{\left(\frac12-\varepsilon\right)}\sum_{|\rho-s|<\frac{|\rho|}{2}}\frac{1}{|\zeta'(\rho)||\rho-s|^2}\frac{|ds|}{|s|^{\frac54+\frac\delta2-5\varepsilon}}+O\left(\frac{1}{\log^2 N}\right),\end{aligned}$$

where we used (4), (5) and the bound $\zeta\left(\frac12\pm\varepsilon\pm it\right)\ll|t|^{2\varepsilon}$ (which is a consequence of the Lindelöf hypothesis). Reversing the order of summation and integration, we have that this is bounded by

$$\begin{aligned}\frac{1}{\log^2 N}\sum_\rho\frac{1}{|\zeta'(\rho)|}\int_{\left(\frac12-\varepsilon\right)+i\left(\Im(\rho)-\frac{|\rho|}{2}\right)}^{\left(\frac12-\varepsilon\right)+i\left(\Im(\rho)+\frac{|\rho|}{2}\right)}\frac{|ds|}{|\rho-s|^2|s|^{\frac54+\frac\delta2-5\varepsilon}}+O\left(\frac{1}{\log^2 N}\right)\\ \ll\frac{1}{\log^2 N}\sum_\rho\frac{1}{|\zeta'(\rho)||\rho|^{\frac54+\frac\delta2-5\varepsilon}}\ll\frac{1}{\log^2 N},\end{aligned}$$

if $\varepsilon<\frac{\delta}{10}$.

Now, by Lemma 3 and the trivial estimate $F_s(z)=O\left(N^{-\frac52}\right)$, all the other terms in (6) are trivially $O\left(\frac{1}{\log^2 N}\right)$ apart from

$$-\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{\left(\frac12-\varepsilon\right)}\frac{\zeta'}{\zeta}(1-s)\sum_\rho R_N(\rho,s)\frac{\zeta(s)}{s(1-s)}\,ds.\tag{7}$$

The integrand has a double pole at every zero ρ of residue

$$\begin{aligned}\operatorname{Res}_{s=\rho}\left(\frac{\zeta'}{\zeta}(1-s)\sum_\rho R_N(\rho,s)\frac{\zeta(s)}{s(1-s)}\right)&=\frac{\log N-\frac12\frac{\zeta''(\rho)}{\zeta'(\rho)}+\frac{\chi'}{\chi}(\rho)+\frac{1-2\rho}{|\rho|^2}}{|\rho|^2}\\ &=\frac{\log N}{|\rho|^2}+O\left(\frac{1}{|\rho|^{2-\varepsilon}|\zeta'(\rho)|}+\frac{1}{|\rho|^2}\right),\end{aligned}$$

where we used the bound $\zeta''\left(\frac12+it\right)\ll|t|^\varepsilon$, which follows from the Lindelöf hypothesis and Cauchy's estimate for the derivatives of a holomorphic function. It follows that moving the line of integration in (7) to $\Re(s) = \frac{1}{2} + \varepsilon$ we get that the integral is equal to

$$\frac{1}{\log N} \sum_{\rho} \frac{1}{|\rho|^2} + O\left(\frac{1}{\log^2 N}\right),$$

and Theorem 1 then follows. ☐

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SANDRO BETTIN – SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, QUEENS AVE, BRISTOL, BS8 1SN, UK

Current address: Centre de Recherches Mathématiques - Université de Montral, P.O. Box 6128, Centre-ville Station, Montréal (Québec), H3C 3J7, Canada

J. BRIAN CONREY – AMERICAN INSTITUTE OF MATHEMATICS, 360 PORTAGE AVENUE, PALO ALTO, CA 94306 - 2244 USA

DAVID W. FARMER – AMERICAN INSTITUTE OF MATHEMATICS, 360 PORTAGE AVENUE, PALO ALTO, CA 94306 - 2244 USA