MOMENTS, PERIOD FUNCTIONS AND COTANGENT SUMS

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Abstract

This thesis is divided into three parts.

In the first part we study the uniformity in the shifts in the asymptotic formulae for the second moment of the Riemann zeta-function and the first moments of the Hecke and the quadratic Dirichlet L-functions.

In the second part we investigate the period function of the Eisenstein series. We use our results to give a simple proof of the Voronoi formula and to prove an exact formula for the second moment of the Riemann zeta function. Moreover, we study a family of cotangent sums, functions defined over the rationals, that generalize the Dedekind sum and share with it the property of satisfying a reciprocity formula.

In the third part, we find optimal Dirichlet polynomials for the Nyman-Beurling criterion for the Riemann-hypothesis, conditionally on some separation condition on the zeros of $\zeta(s)$ and on the Riemann hypothesis.

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Notation

We use the symbols O, o, $\sim$, Ω, $\approx$, $\ll$ and $\gg$ in accordance with their standard meaning in analytic number theory (cf. [Dav]).

For a positive integer n and a complex number a, we will write $\sigma_a(n)$ to indicate $\sum_{d|n} d^a$, the sum of the a-th powers of the divisors of n. We also write $d(n)$ for $\sigma_0(n)$. For integers $a_1, \dots a_n$ we denote by $(a_1, \dots, a_n)$ the greatest common divisor of $a_1, \dots, a_n$.

For a complex number z, we write $\mathrm{e}(z)$ for $e^{2\pi i z}$. Moreover we will often use the symbol $\mathbb{C}'$ to indicate the split complex plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

For a real number c, $\int_{(c)} \mathrm{d}z$ is the contour integral taken along the vertical line $\Re(z) = c$ from $c - i\infty$ to $c + i\infty$.

Throughout this thesis ε is any small positive real number on which all implied constants are allowed to depend. Two uses of ε do not imply that the ε refer to the same quantity.

Chapter 1

Introduction

The Riemann zeta-function is defined as the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$. Euler was the first to grasp the importance of this function in the study of prime numbers. In 1737 he noticed that the fundamental theorem of arithmetic implies that $\zeta(s)$ can be expressed as an infinite product over prime numbers,

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \Re(s) > 1.$$

Using this formula, known as Euler's product, Euler could show that the series of the reciprocal of primes diverges,

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty.$$

A century later, Dirichlet started the theory of L-functions, studying some twisted versions of the Riemann zeta-function (the Dirichlet L-functions). He showed that these satisfy an analogue of Euler's product formula and went on to prove that there are infinitely many primes in any admissible arithmetic progression. This was possibly the greatest achievement on prime numbers since Euclid's proof on their infinitude, however the power of $\zeta(s)$ as a tool to study prime numbers was not fully understood until a few years later. In 1859, in his only work on the subject, Riemann studied $\zeta(s)$ as a function of a complex variable, showing it satisfies several astonishing properties. First, he proved that $\zeta(s)$ can be extended to an analytic function in the whole complex plane apart from a simple pole at $s = 1$. Second, he showed that the Riemann zeta-function satisfies a functional equation,

$$\zeta(1-s) = \chi(1-s)\,\zeta(s),$$

which allows us to relate the values of $\zeta(s)$ on the right of the "critical line" $\Re(s) = \frac{1}{2}$ with those on the left by multiplying by a rather well understood function $\chi(s)$. Moreover, he studied the distribution of the zeros of $\zeta(s)$, showing that $\zeta(s)$ has "trivial" zeros at the negative even integers and that all the "non-trivial" zeros are located inside the "critical strip" $0 \leq \Re(s) \leq 1$ and asserted that it is very likely that all these zero actually lie on the center of the strip, on the critical line. This conjecture, known as the Riemann hypothesis, is still unproven and is one of the most important open problems of mathematics. The relevance of the location of the zeros of $\zeta(s)$ is revealed by another discovery of Riemann, an explicit formula that expresses the prime counting function as an infinite series involving the zeros of $\zeta(s)$. In particular, from this explicit formula one gets that the Riemann hypothesis implies that the primes are "as regularly distributed as we can hope for".

Since the work of Riemann, the study of $\zeta(s)$ has been an invaluable tool to comprehend prime numbers and has lead to many substantial results. Among these, we mention the prime number theorem, which asserts that the number of primes up to x is asymptotic to $\frac{x}{\log x}$ and was proved by de la Vallée-Poussin and Hadamard by showing that $\zeta(s)$ has no zero on the line $\Re(s) = 1$.

This thesis is divided into three parts, all of which belong to the theory of the Riemann zeta-function and L-functions.

In the first part we compute the mean values ("moments") of the Riemann zeta-function and other two families of L-functions, concentrating in particular on the uniformity of the asymptotic formulae in some parameter shifts.

In the second part we investigate the properties of the period function of the Eisenstein series. The Eisenstein series $E_a(z)$ is essentially the Mellin transforms of $\zeta(s)\zeta(s-a+1)$ times the Gamma function $\Gamma(s)$ and its period function $\psi_a(z)$ measures the lack of modularity of $E_a(z)$. We study the Taylor series of $\psi_a(z)$ and apply our results, together with the analytic continuation, to many related problems. In particular, we deduce an exact formula for the second moment of $\zeta(s)$ and a surprising reciprocity formula for a family of cotangent sums.

The third part concerns the Nyman-Beurling criterion for the Riemann-hypothesis, a criterion that basically asserts that the Riemann hypothesis is true if and only if $1/\zeta(s)$ can be well approximated on average on the critical line by Dirichlet polynomials. Moreover, it can be reformulated in terms of the cotangent sums studied in the second part. We find optimal Dirichlet polynomials for this criterion, conditional on a separation condition on the zeros of $\zeta(s)$ and, of course, on the Riemann hypothesis.

More details may be found in the introductions of the three parts (Chapters 2 and 6 and Section 13.1).

Part A

Uniformity in shifted moments of L-functions

Chapter 2

Introduction

In analytic number theory one considers an L-function attached to some arithmetic or geometric data (or to an automorphic form) and tries to obtain results on this data by investigating the related L-function. Typically, the properties that one aims to prove on the L-function side are about the location of the zeros or about the values of the function at some special points. A prototypical example is given by the connection between the Riemann zeta-function $\zeta(s)$ and prime numbers, where the best upper bound for the real part of the zeros of $\zeta(s)$ controls the size of the error term in the prime number theorem. Another example is given by the class number formula, which relates the size of the class number of a quadratic field $\mathbb{Q}(\sqrt{d})$, where d is a fundamental discriminant, to the value at 1 of the quadratic Dirichlet L-function associated to the fundamental discriminant d.

In general, it is very difficult to prove results for a specific value of an L-function and one tries to overcome this problem by considering averages as the value varies or as the L-function varies over a "family of L-functions". One can then deduce results for a specific L-function by appealing to tools such as the Cauchy-Schwartz inequality or some positivity results (and perhaps combining these with the use of "mollifiers" and "amplifiers").

To give a simple example, we are unable to show that the Riemann zeta-function $\zeta\left(\frac{1}{2}+it\right)$ grows more slowly than any power of t on the critical line (the "Lindelöf hypothesis"), however we can show that this is true on average for $t \in [T, 2T]$. More precisely we have that

$$\frac{1}{T}\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \sim \log T \tag{2.0.1}$$

as T goes to infinity [HL] (notice that we can think of this integral as an average over the continuous family $\left\{L_t\left(\frac{1}{2}\right) \mid t \in [T, 2T]\right\}$, where $L_t(s) := \zeta(s+it)$). Going deeper, one has that the Lindelöf hypothesis is also true on average over the shorter interval $[T, T+T^{\frac{1}{3}}]$ and one has

$$\frac{1}{T^{\frac{1}{3}}}\int_T^{T+T^{\frac{1}{3}}}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \sim \log T,$$

which implies the non-trivial bound $\zeta\left(\frac{1}{2}+it\right) \ll t^{\frac{1}{6}+\varepsilon}$ for any individual value of t (see [Bal] and [Watt]).

Similarly, it is conjectured that the Hecke L-function $L\left(\frac{1}{2}, f\right)$ is non-zero for all holomorphic cusp forms f of level 1, weight $2k$ and even functional equation. While we are unable to prove this in full generally, we can show that $L\left(\frac{1}{2}, f\right)$ is non-zero at least almost half the time, when averaging over all the Hecke L-functions of level 1 and weight between K and $2K$, as K tends to infinity (see [IS]).

Given these examples, it is no surprise that the study of L-functions as part of families has gained a prominent role in analytic number theory. One of the great discoveries of the late 20th century was to understand that averages and statistics over these families are not as unpredictable as the individual L-functions, but can be anticipated by studying the symmetries of the related families.

The first step in this direction was given by Montgomery [Mon]. His work indicates that the distribution of the differences between zeros (the "pair correlation") of the Riemann zeta-function is the same as that of the (arguments of the) eigenvalues of the matrices in the unitary group $\mathrm{U}(N)$, with respect to the Haar measure (both the zeros and the eigenvalues should be scaled to have unit mean spacing). This result (which is partially conjectural) was confirmed numerically by Odlyzko [Odl] and extended to the 3-point correlation by Hejhal [Hej] and to the n-point correlations by Rudnik and Sarnak [RS], who also extended this calculation to other individual L-functions in the t-aspect (averaging over t as described at (2.0.1)).

Katz and Sarnak [KaS99a, KaS99b] studied the distributions of zeros near the critical value within families of L-functions suggesting that these coincide with the distributions of eigenvalues near 1 of one of the classical compact groups $\mathrm{U}(N)$, $\mathrm{O}(N)$ and $\mathrm{USp}(2N)$ (resp. the unitary, orthogonal and unitary symplectic group), depending on the family of L-functions considered.

The intuition of Katz and Sarnak has been verified in many cases (see, for example [ILS], [OS] and [Rub]) and the study of the distribution of low zeros reveals, for example, that

- $\mathcal{F}_1 := \{L(s,\chi) \mid q \text{ positive integers}, \chi \text{ primitive character modulo } q\}$ forms a unitary family;
- $\mathcal{F}_2 := \{L(s,f) \mid f \in S_2^*(N), N \in 2\mathbb{N}\}$ forms an orthogonal family, where $S_2^*(N)$ indicates the newforms of level N and weight 2);
- $\mathcal{F}_3 := \{L\big(s, \big(\frac{8d}{\cdot}\big)\big) \mid d \geq 1 \text{ odd and square-free}\}$ forms a symplectic family, where $\big(\frac{\cdot}{\cdot}\big)$ is the Kronecker symbol.

The subsequent work of Keating and Snaith [KeS00a] suggests that the classical compact groups give good models not only for the distributions of zeros of the L-functions, but also for the distributions of values. Keating and Snaith considered the problem of computing the asymptotics for the $2k$-th moment of the Riemann zeta-function,

$$I_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}$$

and conjectured that, up to a well understood arithmetic factor, $I_k(T)$ is asymptotic to the $2k$-th moment of the characteristic polynomials of the matrices in $\mathrm{U}(N)$, averaged over the whole unitary group (one chooses $N \approx \log \frac{T}{2\pi}$

so that the average zero spacing equals the average eigenvalue spacing). This conjecture agrees with the known asymptotics for $k = 1, 2$ ([HL], [Ing]) and with the conjectures, based on number-theoretical computations, for $k = 3, 4$ ([CGh], [CGo]).

Keating and Snaith's conjecture has been extended also to moments of families of L-functions and one expects that the k-th moment for a family $\mathcal{F}$ of L-functions is asymptotic to the k-th moment of the characteristic polynomials for matrices in the classical compact group associated to $\mathcal{F}$ (see [CF], [KeS00b], [CFKRS] and, using a completely different approach based on multiple Dirichlet series, [DGH]). The mean value of the characteristic polynomials in the classical compact groups can be computed exactly and, for any family of L-functions $\mathcal{F}$, one expects that

$$\frac{1}{|\mathcal{F}(Q)|} \sum_{\substack{f \in \mathcal{F}, \\ c(f) \leq Q}} \left\langle L\left(\frac{1}{2}, f\right)\right\rangle_{G(\mathcal{F})}^{k} = P_{k,\mathcal{F}}(\log Q) + o(1), \tag{2.0.2}$$

for a polynomial $P_{k,\mathcal{F}}$ of degree k^2, $\frac{k(k-1)}{2}$ or $\frac{k(k+1)}{2}$ according to whether the classical group $G(\mathcal{F})$ associated to $\mathcal{F}$ is $\mathrm{U}(N)$, $\mathrm{O}(N)$ or $\mathrm{USp}(N)$. In (2.0.2) $c(f)$ indicates the analytic conductor of $L(f, s)$ and is such that $\log c(f)$ is approximately the density of the number of zeros of $L(f, s)$ in a bounded domain of the critical strip. Moreover, $|\mathcal{F}(Q)|$ is the number of $f \in \mathcal{F}$ with $c(f) \leq Q$ and $\langle x\rangle_G = x$ if $G = \mathrm{O}(N)$ or $G = \mathrm{USp}(N)$ and $\langle x\rangle_G = |x|^2$ if $G = U(N)$. Notice, in particular, that the difference in the degree of the polynomial $P_{k,\mathcal{F}}$ (as well as the different structure of the leading term) makes the computation of the moments a quick way to determine the symmetry of a family.

The moment conjecture (2.0.2) is very far from being proved and has been verified only for a few values of k for L-functions of low degrees. For example, for the family $\mathcal{F}_1$ defined above, the asymptotic (2.0.2) has been proved only for $k = 1, 2$ (where in the case $k = 2$ the full main term is known only for prime moduli) and $k = 3$ when some extra averaging is introduced (see [Pal], [Sou07], [You11] and [CIS]), whereas for the families $\mathcal{F}_2$ and $\mathcal{F}_3$ we can compute the cases $k = 1, 2, 3, 4$ (see [Duk] and [KMV]) and $k = 1, 2, 3$ respectively (see [Jut81] and [Sou00]).

When investigating the asymptotic behaviour of the moments, it is extremely useful to start by considering the shifted moments

$$\frac{1}{|\mathcal{F}(Q)|} \sum_{\substack{f \in \mathcal{F}, \\ c(f) \leq Q}} L\Big(\frac{1}{2} + \alpha_1, f\Big) \cdots L\Big(\frac{1}{2} + \alpha_k, f\Big).$$

In fact, the random matrix theory model for the unshifted moments leads only to the leading term of the polynomial $P_{k,\mathcal{F}}$ in (2.0.2). However, when the shifts are added the combinatorial structure behind the main terms comes out explicitly and by eventually letting the shifts go to zero one gets also the lower order terms (see [CFKRS]).

It is also natural to consider mean values of ratios of L-functions,

$$\frac{1}{|\mathcal{F}(Q)|} \sum_{\substack{f \in \mathcal{F}, \\ c(f) \leq Q}} \frac{L\big(\frac{1}{2} + \alpha_1, f\big) \cdots L\big(\frac{1}{2} + \alpha_m, f\big)}{L\big(\frac{1}{2} + \beta_1, f\big) \cdots L\big(\frac{1}{2} + \beta_n, f\big)}. \tag{2.0.3}$$

These kinds of averages were first considered by Farmer [Far], who gave a conjecture for the ratio with two Riemann zeta-functions in the numerator and denominator. This conjecture (known as the "ratio conjecture") asserts that

$$\int_0^T \frac{\zeta\big(\frac{1}{2} + it + \alpha_1\big)\zeta\big(\frac{1}{2} + it + \alpha_2\big)}{\zeta\big(\frac{1}{2} + it + \beta_1\big)\zeta\big(\frac{1}{2} + it + \beta_2\big)} \,\mathrm{d}t \sim T \frac{(\alpha_1 + \beta_2)(\alpha_2 + \beta_1)}{(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)} - T^{1-\alpha_1-\alpha_2} \frac{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}{(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)},$$

for $|\Re(\alpha_j)| < \frac{1}{4}$, $\frac{1}{\log T} < \Re(\beta_j) < \frac{1}{4}$ and $\Im(\alpha_j), \Im(\beta_j) \ll T^{1-\varepsilon}$ with $j = 1, 2$. Conrey, Farmer, and Zirnbauer extended this to other families of L-functions [CFZ], conjecturing that the averages (2.0.3) have the same structure of the analogous ratios of characteristic polynomials in the random matrix setting. The ratios conjectures have not been proved in any case with at least one L-function in the denominator, but so far all the results that can be deduced from them always agree with what is known and have been checked numerically in some cases (see, for example, [HKS]).

The ratios conjectures have proved to be a very useful tool to study many problems related to the distribution of zeros of L-functions and give, for example, a quick way to produce conjectures (with lower order terms) for "mollified" moments or n-point correlations of families of L-functions (see [CS]). These applications are usually obtained by integrating the ratios conjectures over the shifts on long vertical intervals and therefore one needs to assume that these conjectures are true uniformly for shifts that have large imaginary parts. The random matrix models do not help us definitively to understand in what range of parameter shifts the moments and ratios conjectures are uniform. In fact, the averages analogous to (2.0.3) for the classical compact groups can be computed exactly with no error terms [BS] and thus are automatically completely uniform in the shifts. Similarly the "recipe approach" [CFKRS], which essentially produces conjectures by substituting the L-functions with their approximate functional equations and considering only the diagonal terms, does not give any insight into the expected range of uniformity in the shifts.

Part 1 of this thesis is dedicated to providing evidence towards a large uniformity in the shifts of the moments (and indirectly ratios) conjectures. We shall consider three cases, one for each symmetry group. First, we consider a unitary example and examine the uniformity in the second moment for the Riemann zeta-function, verifying that the shifted (continuous) analogue of (2.0.2) holds for parameters that can be as large as $Q^{2-\varepsilon}$ (we see also that under the Lindelöf hypothesis one of the two parameters can be as large as any power of Q). Second, we consider the first moment of the orthogonal family consisting of Hecke L-function of weight 2, obtaining the uniformity for shifts α in the range $|\Im(\alpha)| \ll Q^{1-\varepsilon}$. Third, we consider the first moment of the symplectic family consisting of quadratic Dirichlet L-functions and obtain the uniformity for a shift up to $Q^{\frac{3}{5}-\varepsilon}$.

Chapter 3

The second shifted moment of the Riemann zeta-function

The work presented in this chapter was first published in [Bet10].

3.1 Introduction

An important problem in analytic number theory is to understand the moments of the Riemann zeta-function

$$I_k(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \mathrm{d}t.$$

The knowledge of the asymptotic behavior of $I_k(T)$ would give important information about the maximal order of the Riemann zeta function on the critical line and about the zeros of this function and therefore about the distribution of prime numbers. Unfortunately, the asymptotic is known just for $k = 1$ and $k = 2$. Specifically, in 1918 Hardy and Littlewood [HL] proved

$$I_1(T) \sim T \log T \tag{3.1.1}$$

and in 1926 Ingham [Ing] proved

$$I_2(T) \sim \frac{1}{2\pi^2} T \log^4 T.$$

For other k the problem is still open and it is conjectured that

$$I_k(T) = TP_k(\log T) + E_k(T), \tag{3.1.2}$$

where P_k is a polynomial of degree k^2 and $E_k(T) = o(T)$ (see [CGh], [KeS00a], [CGo], [DGH] and [CFKRS]).

As mentioned above, results on moments of the Riemann zeta-function can be used to obtain upper bounds for the growth of $\zeta(s)$ in the critical line. This is displayed clearly by the following result of Heath-Brown (see (7.20.2) in [Tit])

$$\zeta\left(\frac{1}{2} + iT\right)^{2k} \ll (\log T)\left(1 + \frac{1}{2\pi}\int_{T-\log^2 T}^{T+\log^2 T}\left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \mathrm{d}t\right), \tag{3.1.3}$$

which shows, for example, that the crude bound $I_k(T) \ll T^A$ for infinitely many natural numbers k and some fixed $A > 1$ implies the Lindelöf hypothesis, i.e. implies that $\zeta\big(\frac{1}{2} + it\big) \ll |t|^{\varepsilon}$ for all $\varepsilon > 0$ (and $|t| > 1$).

Moreover, Heath-Brown's result has important consequences if we keep k fixed. For example, taking $k = 1$ and denoting by ϑ the smallest real number such that $E_1(T) \ll T^{\vartheta+\varepsilon}$, one has that

$$\zeta\left(\frac{1}{2} + it\right) \ll |t|^{\frac{\vartheta}{2}+\varepsilon} \qquad (|t| > 1), \tag{3.1.4}$$

for all $\varepsilon > 0$. However, Good [Goo] showed that $E_1(T) = \Omega(T^{\frac{1}{4}})$ and thus one can not prove anything better than $\zeta\big(\frac{1}{2} + it\big) \ll |t|^{\frac{1}{8}+\varepsilon}$ just by giving upper bounds for $E_1(T)$.

This problem can be overcome by considering integrals of the Riemann zeta-function over short intervals or, more generally, moments with high shifts. In fact, if we could prove that

$$I(T, a, b) := \int_0^T \zeta\left(\frac{1}{2} + a + it\right)\zeta\left(\frac{1}{2} - b - it\right) \mathrm{d}t$$

is bounded by $T^{1+\varepsilon}$ for all purely imaginary numbers a, b bounded by any fixed power of T, then the Lindelöf hypothesis would follow.

In [Ing], Ingham computed the asymptotics of $I(T,a,b)$ for all bounded numbers a and b (where one has to exclude a small interval from the integral if the line of integration is close to a pole of the Riemann zeta functions). He showed that

$$I(T,a,b) \sim \int_1^T \left(\zeta(1+c) + \left(\frac{t}{2\pi}\right)^{-c} \zeta(1-c) \right) \mathrm{d}\tau,$$

where $c = a - b$, with an error term of size $O\Big(T^{\frac{1}{2}+\varepsilon}\Big)$ if $\Re(a), \Re(b) \ll \frac{1}{\log T}$.

In this chapter we extend the work of Ingham allowing the imaginary part of the shifts a and b to grow with T. For the sake of convenience, we assume that the real parts of a and b are close to 0, but the same method should work also for bounded $\Re(a)$ and $\Re(b)$. More specifically, we obtain the following.

Theorem 3.1.1. *Let $T \geq 2$, and let $a, b, \in \mathbb{C}$ be such that*

$$\Re(a) \ll \frac{1}{\log T}, \qquad \Re(b) \ll \frac{1}{\log T}.$$

Moreover, let u, v be the positive real numbers defined by

$$\begin{aligned} T^u &= \max(|\Im(a)|, |\Im(b)|) + T, \\ T^v &= \min(|\Im(a)|, |\Im(b)|) + T \end{aligned} \tag{3.1.5}$$

and assume $u, v \leq A$ for some fixed $A > 0$. Let $r > 0$ be such that

$$\zeta\left(\frac{1}{2} + it\right) \ll 1 + |t|^r.$$

Then, writing $c = a - b$, for all $\varepsilon > 0$ we have

$$\begin{aligned} &\int_0^T \zeta\left(\frac{1}{2} + a + it\right) \zeta\left(\frac{1}{2} - b - it\right) \mathrm{d}t = \\ &\quad = \int_0^T \left(\zeta(1+c) + \zeta(1-c)\, \chi\left(\frac{1}{2} + a + it\right) \chi\left(\frac{1}{2} - b - it\right) \right) \mathrm{d}t + \\ &\qquad + O\Big(\min\big(T^{\frac{v}{2}+ru+\varepsilon}, T^{\frac{u}{2}} \log^2 T\big)\Big), \end{aligned} \tag{3.1.6}$$

as $T \to \infty$. If $c = 0$, then the integrand on the right hand side of (3.1.6) *has to be interpreted as the pointwise limit for $c \to 0$.*

Remark 3.1.2. *In the Theorem we can take any* r *greater than* $\frac{32}{205} \approx 0.15609\ldots$ *(Huxley, [Hux]).*

The χ function in the theorem is defined as

$$\chi(1-s) := 2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}$$

and is the function that appears in the functional equation for the Riemann zeta-function,

$$\zeta(1-s) = \chi(1-s)\zeta(s). \tag{3.1.7}$$

We remark that the factor $\chi\big(\frac{1}{2}+a+it\big)\chi\big(\frac{1}{2}-b-it\big)$ in the statement of the theorem can be replaced by $\Big(\frac{|t+\Im(b)|}{2\pi}\Big)^{-c}$ (or, equivalently, by $\Big(\frac{|t+\Im(a)|}{2\pi}\Big)^{-c}$) as will be clear from the proof of the theorem. However, in general this factor can not be replaced by $\big(\frac{t}{2\pi}\big)^{-c}$, since

$$\int_1^T \left(\left(\frac{|t+\Im(a)|}{2\pi}\right)^{-c} - \left(\frac{t}{2\pi}\right)^{-c}\right)\mathrm{d}t$$

is larger than the error term if, for example, $T^{\frac{1}{2}+\varepsilon} \ll |a| \ll (T/|c|)^{1-\varepsilon}$ and $c \gg T^{\varepsilon}$.

The main ideas of the proof of Theorem 3.1.1 stem from the proof of Theorem 7.4 in [Tit] about the second moment of the Riemann Zeta function without shifts.

3.2 The shifted divisor problem

Remark 3.2.1. *In the whole chapter we consider* a*,* b *and* $c = a - b$ *to be complex numbers and we write*

$$a = \alpha + i\alpha',$$

$$b = \beta + i\beta',$$

$$c = \gamma + i\gamma',$$

with $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in \mathbb{R}$.

In this section we compute the asymptotics for

$$D_c(x) = \sum_{mn\le x} \frac{1}{n^c} = \sum_{n\le x} \sigma_{-c}(n),$$

where $\sigma_\eta(n) := \sum_{d|n} d^\eta$, most interestingly in the case when $\Im(c)$ is large, as the asymptotics is well known for bounded c.

Lemma 3.2.2. *Let $x \ge 2$ and assume that*

$$|\gamma| \ll \frac{1}{\log x}.$$

Moreover, let r be such that

$$\zeta\left(\frac{1}{2} + it\right) \ll 1 + |t|^r.$$

Then, for any $\varepsilon > 0$, we have

$$\begin{aligned} D_c(x) &= \zeta(1+c)\, x + \zeta(1-c)\frac{x^{1-c}}{1-c} + E_c(x) \\ &= \int_1^x \big(\zeta(1+c) + \zeta(1-c)\, u^{-c}\big)\, \mathrm{d}u + E_c(x), \end{aligned} \tag{3.2.1}$$

where

$$E_c(x) \ll \min\Big(x^{\frac{1}{3}+\varepsilon} + |\gamma'|^{\frac{1}{2}} \log^2 x, x^{\frac{1}{2}+\varepsilon}(|\gamma'|+1)^r\Big),$$

for all $\varepsilon > 0$. As above, if $c = 0$ the right hand side of (3.2.1) *has to be interpreted as the limit for $c \to 0$.*

Proof. Clearly we may assume that x is half an odd integer.

Let $Q \ge 1$, $\varepsilon > 0$ and $\eta = \frac{1}{\log x} + \max(1, 1-\gamma)$. Applying Lemma 3.12 in [Tit] with $a_n = \sigma_{-c}(n)$, $\psi(n) = n^\varepsilon$ and with the c of the lemma equal to η, we find that

$$D_c(x) = \frac{1}{2\pi i}\int_{\eta-iQ}^{\eta+iQ} \zeta(s)\,\zeta(s+c)\frac{x^s}{s}\,\mathrm{d}s + O\left(\frac{x^{1+\varepsilon}\log x}{Q}\right), \tag{3.2.2}$$

since for $\Re(s) > \max(1, 1-\gamma)$ one has

$$\zeta(s)\,\zeta(s+c) = \sum_{n\ge1} \frac{\sigma_{-c}(s)}{n^s}$$

(see, for example, (1.3.1) in [Tit]).

Taking $Q = x$ (and replacing Q with $Q+1$ if $\big|x - |\gamma'|\big| < \frac{1}{2}$ in order to avoid coming close to the pole of $\zeta(s+c)$), one has that

$$\zeta(s)\,\zeta(s+c) = O\Big(x^{r-\frac{1}{2}}(|\gamma'|+x)^r\Big)$$

if $\frac{1}{2} \le \Re(s) \le 2$ and $\Im(s) = \pm Q$. Thus, from (3.2.2) by contour integration we have that

$$\begin{aligned} D_c(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-iQ}^{\frac{1}{2}+iQ} \zeta(s)\zeta(s+c)\frac{x^s}{s}\,\mathrm{d}s + \zeta(1+c)\,x + \\ + \zeta(1-c)\frac{x^{1-c}}{1-c} + O\Big(x^{r-\frac{1}{2}}(|\gamma'|+x)^r + x^{\varepsilon}\Big). \end{aligned}$$

By the Cauchy-Schwartz inequality we have that

$$\begin{aligned} \left(\int_{\frac{1}{2}-iQ}^{\frac{1}{2}+iQ} \zeta(s)\zeta(s+c)\frac{x^s}{s}\,\mathrm{d}s\right)^2 &\ll x\int_{-Q}^{Q}\frac{|\zeta\big(\frac{1}{2}+it\big)|^2}{|t|+1}\,\mathrm{d}t\int_{-Q}^{Q}\frac{|\zeta\big(\frac{1}{2}+c+it\big)|^2}{|t|+1}\,\mathrm{d}t \\ &\ll x^{1+\varepsilon}(|\gamma'|+x)^{2r}, \end{aligned}$$

by (3.1.1) and partial integration. Thus,

$$E_c(x) \ll x^{\frac{1}{2}+\varepsilon}(|\gamma'|+x)^r. \tag{3.2.3}$$

Now, we go back to (3.2.2) and this time we choose $Q = x^{\frac{2}{3}}$ (again replacing Q with $Q+1$ if necessary) and move the path of integration to $\nu = -\frac{1}{\log x} + \min(0, -\gamma)$. Since the integrand is $O\Big(x^{\frac{1}{3}}\log^2 x + |\gamma'|^{\frac{1}{2}}x^{-\frac{1}{3}}\log(2+|\gamma'|)\Big)$ on the horizontal lines, we find

$$\begin{aligned} \frac{1}{2\pi i}\int_{\eta-iQ}^{\eta+iQ}\zeta(s)\zeta(s+c)\frac{x^s}{s}\,\mathrm{d}s = \frac{1}{2\pi i}\int_{\nu-iQ}^{\nu+iQ}\zeta(s)\zeta(s+c)\frac{x^s}{s}\,\mathrm{d}s + \zeta(1+c)x + \\ + \zeta(1-c)\frac{x^{1-c}}{1-c} + \zeta(c) + \\ + O\Big(x^{\frac{1}{3}}\log^2 x + |\gamma'|^{\frac{1}{2}}x^{-\frac{1}{3}}\log(2+|\gamma'|)\Big), \end{aligned} \tag{3.2.4}$$

where $\zeta(c)$ has to be omitted if $|\gamma'| \ge x^{\frac{2}{3}} + \frac{1}{2}$ and so it can always be inserted in the error term, since

$$\zeta(c) \ll |c|^{\frac{1}{2}}\log(|c|+2),$$

by Theorem 3.5 in [Tit] and the functional equation (3.1.7). Moreover, applying the functional equation and expanding the product of zetas as its Dirichlet series, we find

$$\begin{aligned}\int_{\nu-iQ}^{\nu+iQ}\zeta(s)\zeta(s+c)\frac{x^s}{s}\,\mathrm{d}s &= \int_{\nu-iQ}^{\nu+iQ}\chi(s)\chi(s+c)\zeta(1-s)\zeta(1-s-c)\frac{x^s}{s}\,\mathrm{d}s\\ &= ix^{\nu}\sum_{n=1}^{\infty}\frac{\sigma_{-c}(n)}{n^{1-c-\nu}}\int_{-Q}^{Q}\chi(\nu+it)\chi(\nu+c+it)\frac{(xn)^{it}}{\nu+it}\,\mathrm{d}t.\end{aligned} \tag{3.2.5}$$

The asymptotic expansion in a strip for $\chi(s)$ (see, for example, (4.12.3) in [Tit]) gives

$$\begin{aligned}&i\chi(\nu+it)\chi(\nu+c+it)\frac{(xn)^{it}}{\nu+it} =\\ &\qquad= \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\nu-it}\left(\frac{|t+\gamma'|}{2\pi}\right)^{\frac{1}{2}-\nu-c-it}\times\\ &\qquad\quad\times e^{i\left(2t+\gamma'+\frac{\pi}{4}(\operatorname{sgn}(t)+\operatorname{sgn}(t+\gamma'))\right)}\frac{(xn)^{it}}{t}\left(1+O\left(\frac{1}{|t|}+\frac{1}{|t+\gamma'|}\right)\right)\\ &\qquad= g(t)\,e^{ih(t)}\left(1+O\left(\frac{1}{|t|}+\frac{1}{|t+\gamma'|}\right)\right),\end{aligned}$$

where

$$\begin{aligned}g(t) &= \frac{1}{t}\left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\nu}\left(\frac{|t+\gamma'|}{2\pi}\right)^{\frac{1}{2}-\nu-\gamma},\\ h(t) &= -t\log\left(\frac{|t|}{2\pi}\right)-(\gamma'+t)\log\left(\frac{|t+\gamma'|}{2\pi}\right)+2t+\gamma'+\\ &\quad+\frac{\pi}{4}(\operatorname{sgn}(t)+\operatorname{sgn}(t+\gamma'))+t\log(xn)\end{aligned} \tag{3.2.6}$$

and thus

$$\begin{aligned}h'(t) &= -\log\left(\frac{|t|}{2\pi}\right)-\log\left(\frac{|t+\gamma'|}{2\pi}\right)+\log(xn) = -\log\left(\frac{|t+\gamma'||t|}{(2\pi)^2xn}\right),\\ h''(t) &= -\frac{2t+\gamma'}{t(t+\gamma')}.\end{aligned}$$

Here and throughout the rest of the proof we are implicitly removing two intervals of length 1 around $t=0$ and $t=-\gamma'$, whose contribution to (3.2.5) is $O\left(|\gamma'|^{\frac{1}{2}}+1\right)$.

Now, let's consider the case $|\gamma'| < 3Q$. If $n > \frac{Q^2}{x}$, we have trivially that $h' \gg 1$. If $n \leq \frac{Q^2}{x}$, defining

$$\begin{aligned} V &= \Big\{ t \mid \big| |t+\gamma'||t| - (2\pi)^2 xn \big| < \sqrt{n} x^{\frac{5}{6}} \Big\}, \\ U &= [-Q, Q] \setminus V, \end{aligned}$$

we have that the the measure of V is $\mathrm{m}(V) \ll x^{\frac{1}{3}}$ and, for $t \in U$, we have

$$h'(t) \gg \frac{1}{x^{\frac{1}{6}} n^{\frac{1}{2}}}.$$

Thus, using Lemma 4.3 in [Tit], we have that

$$\begin{aligned} ix^{\nu} \sum_{n=1}^{\infty} \frac{\sigma_{-c}(n)}{n^{1-c-\nu}} \int_{-Q}^{Q} \chi(\nu+it)\chi(\nu+c+it) \frac{(xn)^{it}}{\nu+it} \,\mathrm{d}t = \\ &= x^{\nu} \sum_{n \leq \frac{Q^2}{x}} \frac{\sigma_{-c}(n)}{n^{1-c-\nu}} \int_{-Q}^{Q} g(t) e^{ih(t)} \,\mathrm{d}t + O\big(\log^2 x\big) \\ &= x^{\nu} \sum_{n \leq \frac{Q^2}{x}} \frac{\sigma_{-c}(n)}{n^{1-c-\nu}} \int_{U} g(t) e^{ih(t)} \,\mathrm{d}t + O\Big(x^{\frac{1}{3}} \log^2 x\Big) \\ &\ll x^{\frac{1}{3}} \log^2 x, \end{aligned} \tag{3.2.7}$$

if $|\gamma'| < 3Q$. Now, if $|\gamma'| \geq 3Q$, we have

$$h''(t) \gg \frac{1}{Q},$$

in $[-Q, Q]$. Thus, using Lemma 4.5 in [Tit], we have

$$ix^{\nu} \sum_{n=1}^{\infty} \frac{\sigma_{-c}(n)}{n^{1-c-\nu}} \int_{-Q}^{Q} \chi(\nu+it)\chi(\nu+c+it) \frac{(xn)^{it}}{\nu+it} \,\mathrm{d}t \ll |\gamma'|^{\frac{1}{2}} \log^2 x.$$

This equation, together with (3.2.2)-(3.2.7), implies the stated result. □

3.3 Proof of Theorem 3.1.1

We start with the following asymptotics for $\chi\big(\frac{1}{2}+a+it\big)\chi\big(\frac{1}{2}-b-it\big)$.

Lemma 3.3.1. *Let* $|\alpha|, |\beta| \ll 1$ *and let*

$$\begin{aligned} |t+\alpha'| &> 10|c|, \\ |t+\beta'| &> 10|c|. \end{aligned} \tag{3.3.1}$$

Then

$$\chi\Big(\frac{1}{2}+a+it\Big)\chi\Big(\frac{1}{2}-b-it\Big) = \\ e^{-c\log\frac{|t+\beta'|}{2\pi}+(a+it)\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)+c}\left(1+O\Big(\frac{|c|}{|t+\alpha'|}\Big)\right).$$

Proof. Firstly, we remark that (3.3.1) implies

$$|t+\beta'| \le |t+\alpha'| + |c| \ll |t+\alpha'|.$$

Now, since $\chi(s)\chi(1-s)=1$, we have

$$\begin{aligned}\chi\Big(\frac{1}{2}+a+it\Big)\chi\Big(\frac{1}{2}-b-it\Big) &= \frac{\chi\big(\frac{1}{2}+a+it\big)}{\chi\big(\frac{1}{2}+b+it\big)} \\ &= (2\pi)^c\frac{\cos\big(\frac{\pi}{2}\big(\frac{1}{2}-a-it\big)\big)}{\cos\big(\frac{\pi}{2}\big(\frac{1}{2}-b-it\big)\big)}\frac{\Gamma\big(\frac{1}{2}-a-it\big)}{\Gamma\big(\frac{1}{2}-b-it\big)}.\end{aligned} \tag{3.3.2}$$

Stirling's formula, as expressed in (21.1) of [Rad], states

$$\log\Gamma(s) = \Big(s-\frac{1}{2}\Big)\log s - s + \frac{1}{2}\log 2\pi + R(s),$$

where

$$R(s) = \int_0^{+\infty}\frac{g(x)}{(x+s)^2}\,\mathrm{d}x$$

and $g(x)=\{x\}(\{x\}-1)/2$. Therefore

$$\frac{\Gamma\big(\frac{1}{2}-a-it\big)}{\Gamma\big(\frac{1}{2}-b-it\big)} = e^{-(a+it)\log\left(\frac{1}{2}-a-it\right)+(b+it)\log\left(\frac{1}{2}-b-it\right)+c+\Delta_{a,b}(t)},$$

where

$$\Delta_{a,b}(t) = R\Big(\frac{1}{2}-a-it\Big) - R\Big(\frac{1}{2}-b-it\Big).$$

Thus

$$\begin{aligned}\frac{\Gamma\left(\frac{1}{2}-a-it\right)}{\Gamma\left(\frac{1}{2}-b-it\right)} =& \exp\left((a+it)\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)-c\log\left(\frac{1}{2}-b-it\right)+\right.\\ &\left.+c+\Delta_{a,b}(t)\right)\\ =& \exp\left((a+it)\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)+c-c\log|t+\beta'|+\right.\\ &\left.+\Delta_{a,b}(t)-\frac{\pi ic}{2}\operatorname{sgn}(t+\beta')+O\left(\frac{|c|}{|\alpha'+t|}\right)\right).\end{aligned} \tag{3.3.3}$$

Moreover,

$$\begin{aligned}\frac{\cos\left(\frac{\pi}{2}\left(\frac{1}{2}-a-it\right)\right)}{\cos\left(\frac{\pi}{2}\left(\frac{1}{2}-b-it\right)\right)} &= \frac{\cos\left(\frac{\pi}{2}\left(\frac{1}{2}-b-it\right)-\frac{\pi c}{2}\right)}{\cos\left(\frac{\pi}{2}\left(\frac{1}{2}-b-it\right)\right)}\\ &= \cos\frac{\pi c}{2}+\sin\frac{\pi c}{2}\tan\left(\frac{\pi}{2}\left(\frac{1}{2}-b-it\right)\right)\\ &= \cos\frac{\pi c}{2}+\sin\frac{\pi c}{2}\left(i\operatorname{sgn}(t+\beta')+O\left(e^{-\pi|t+\beta'|}\right)\right)\\ &= e^{\frac{\pi ic}{2}\operatorname{sgn}(t+\beta')}+O\left(\left|\sin\left(\frac{\pi c}{2}\right)\right|e^{-\pi|t+\beta'|}\right)\\ &= e^{\frac{\pi ic}{2}\operatorname{sgn}(t+\beta')}+O\left(|c|e^{-\pi|t+\beta'|+\frac{\pi}{2}|c|}\right)\\ &= e^{\frac{\pi ic}{2}\operatorname{sgn}(t+\beta')}+O\left(\frac{|c|}{|t+\alpha'|}\right).\end{aligned} \tag{3.3.4}$$

Finally,

$$\begin{aligned}|\Delta_{a,b}(t)| &= \left|\int_0^{+\infty}\frac{g(x)}{(x+\frac{1}{2}-a-it)^2}-\frac{g(x)}{(x+\frac{1}{2}-b-it)^2}\,\mathrm{d}x\right|\\ &\ll \int_0^{+\infty}\left|\frac{1}{(x+\frac{1}{2}-a-it)^2}-\frac{1}{(x+\frac{1}{2}-b-it)^2}\right|\mathrm{d}x\\ &\ll \int_0^{+\infty}\left|\frac{c^2+2c\left(x+\frac{1}{2}-a-it\right)}{(x+\frac{1}{2}-a-it)^2(x+\frac{1}{2}-b-it)^2}\right|\mathrm{d}x\\ &\ll \frac{|c|^2+2|c||t+\alpha'|}{|t+\alpha'|^2}\int_0^{+\infty}\frac{1}{\left|x+\frac{1}{2}-b-it\right|^2}\,\mathrm{d}x\\ &\ll \frac{|c|^2+2|c||t+\alpha'|}{|t+\alpha'|^2(|t+\alpha'|+1|)}.\end{aligned}$$

This equation, together with (3.3.2)-(3.3.4), gives the result. □

Proof of Theorem 1. By taking the conjugate in (3.1.6) if necessary we may assume that $|\alpha'| \geq |\beta'|$. We remark that with this assumption the numbers u and v defined in (3.1.5) satisfy

$$\begin{aligned} T^u &= |\alpha'| + T, \\ T^v &= |\beta'| + T. \end{aligned}$$

By the functional equation (3.1.7) we have

$$\begin{aligned} I(T,a,b) &= -i \int_{\frac{1}{2}}^{\frac{1}{2}+iT} \zeta(s+a)\,\zeta(1-s-b)\,\mathrm{d}s \\ &= -i \int_{\frac{1}{2}}^{\frac{1}{2}+iT} \chi(1-s-b)\,\zeta(s+a)\,\zeta(s+b)\,\mathrm{d}s. \end{aligned}$$

Moving the path of integration to $\delta = \frac{1}{\log T} + \max(1-\alpha, 1-\beta)$, we obtain

$$\begin{aligned} I(T,a,b) = -i \int_{\delta}^{\delta+iT} \chi(1-s-b)\,\zeta(s+a)\,\zeta(s+b)\,\mathrm{d}s+ \\ + O\big(T^{r(u+v)} + T^{\frac{v}{2}} \log^2 T\big). \end{aligned} \tag{3.3.5}$$

Now, we have

$$\begin{aligned} \int_{\delta}^{\delta+iT} \chi(1-s-b)\,\zeta(s+a)\,\zeta(s+b)\,\mathrm{d}s &= \int_{\delta+b}^{\delta+b+iT} \chi(1-s)\,\zeta(s+c)\,\zeta(s)\,\mathrm{d}s \\ &= \sum_{n,m} \frac{1}{n^c} \int_{\delta+b}^{\delta+b+iT} \chi(1-s)(nm)^{-s}\,\mathrm{d}s \end{aligned} \tag{3.3.6}$$

and, if $\beta' > 0$, by (7.4.2) and (7.4.3) in [Tit], this is

$$\begin{aligned} &\sum_{n,m} \frac{1}{n^c} \int_{\delta+b}^{\delta+b+iT} \chi(1-s)(nm)^{-s}\,\mathrm{d}s \\ &\qquad = 2\pi i \sum_{\frac{\beta'}{2\pi} < nm \leq \frac{T+\beta'}{2\pi}} \frac{1}{n^c} + O\big(T^{\frac{v}{2}}\big) + O(\mathcal{E}(T,a,b)), \end{aligned} \tag{3.3.7}$$

where

$$\begin{aligned} \mathcal{E}(T,a,b) &= \sum_{\substack{|2\pi mn - T| > \frac{1}{2}, \\ |2\pi mn - T - \beta'| > \frac{1}{2}}} \frac{1}{mn} \frac{T^{\frac{v}{2}}}{\min\Big(1, \Big|\log \frac{|\beta'|}{2\pi mn}\Big|, \Big|\log \frac{|T+\beta'|}{2\pi mn}\Big|\Big)} \\ &\ll T^{\frac{v}{2}} \log^2 T, \end{aligned} \tag{3.3.8}$$

since

$$\sum_{|mn-Q|>\frac{1}{2}} \frac{1}{\left|\log\frac{Q}{mn}\right|mn} \ll \sum_{m,n}\frac{1}{mn} + \sum_{\substack{|mn-Q|>\frac{1}{2},\\ \frac{Q}{2m}<n<\frac{2Q}{m}}} \frac{1}{\left|\log\frac{Q}{mn}\right|mn}$$
$$\ll \log^2 Q + \sum_{m,r\ll Q}\frac{1}{mr} \ll \log^2 Q.$$

If $\beta' \leq 0$ we can proceed in the same way, but we have to replace

$$\sum_{\frac{\beta'}{2\pi}<nm\leq\frac{T+\beta'}{2\pi}} \frac{1}{n^c}$$

with

$$\begin{cases} \sum_{0<nm\leq\frac{T+\beta'}{2\pi}} \frac{1}{n^c} + \sum_{0<nm\leq\frac{-\beta'}{2\pi}} \frac{1}{n^c} & \text{if } -T\leq\beta'\leq 0, \\ \sum_{-\frac{T+\beta'}{2\pi}\leq nm<\frac{-\beta'}{2\pi}} \frac{1}{n^c} & \text{if } \beta'\leq -T. \end{cases} \tag{3.3.9}$$

Therefore, by Lemma 3.2.2 and (3.3.5)-(3.3.8), we have

$$\begin{aligned} I(T,a,b) &= 2\pi\int_{\frac{\beta'}{2\pi}}^{\frac{T+\beta'}{2\pi}} \left(\zeta(1+c)+\zeta(1-c)|x|^{-c}\right)\mathrm{d}x+ \\ &\quad + O\Big(\min\Big(|\gamma'|^{\frac{1}{2}}\log^2 T, T^{\frac{v}{2}}|\gamma'|^r\Big) + T^{\frac{v}{2}}\log^2 T\Big), \\ &= \int_0^T \left(\zeta(1+c)+\zeta(1-c)\left(\frac{|t+\beta'|}{2\pi}\right)^{-c}\right)\mathrm{d}t+ \\ &\quad + O\big(\min\big(T^{\frac{u}{2}}\log^2 T, T^{\frac{v}{2}+ru}\big) + T^{\frac{v}{2}}\log^2 T\big), \end{aligned} \tag{3.3.10}$$

since $|\gamma'| \ll T^v$.

Now, if $\gamma' \gg 1$, we have that

$$\int_0^T \chi\Big(\frac{1}{2}+a+it\Big)\chi\Big(\frac{1}{2}-b-it\Big)\,\mathrm{d}t \ll \frac{T}{|\gamma'|},$$

since we can bound trivially the two intervals of length $\frac{T}{|\gamma'|}$ around $t=0$ and $t=-\gamma'$ (if it is inside the domain of integration) and apply Lemma 4.3 of [Tit] to bound the remaining part of the integral (the function $g(t)$ is $O(1)$ and the function $h(t)$ is the same as in (3.2.6) with $xn=1$). Moreover, clearly we have

$$\int_0^T \left(\frac{|t+\beta'|}{2\pi}\right)^{-c}\mathrm{d}t \ll \frac{T^v}{|\gamma'|}.$$

Therefore, if $|\gamma'| > \frac{T^{\frac{v}{2}}}{\log^2 T}$, we have

$$\int_0^T \chi\Big(\frac{1}{2}+a+it\Big)\chi\Big(\frac{1}{2}-b-it\Big)\,\mathrm{d}t = \int_0^T \left(\frac{|t+\beta'|}{2\pi}\right)^{-c}\mathrm{d}t + O\big(T^{\frac{v}{2}}\log^2 T\big).$$

Finally, assume $|\gamma'| \le \frac{T^{\frac{v}{2}}}{\log^2 T}$ (and thus we have also $\gamma' \ll \frac{T^{\frac{u}{2}}}{\log^2 T}$) and let

$$\begin{aligned} W_1 &= [0,T] \cap \big\{t \mid |t+\alpha'|, |t+\beta'| > 10|c|\big\},\\ W_2 &= [0,T] \cap \big\{t \mid |t+\alpha'|, |t+\beta'| \le 10|c|\big\}. \end{aligned}$$

We have

$$\int_{W_2} \chi\Big(\frac{1}{2}+a+it\Big)\chi\Big(\frac{1}{2}-b-it\Big)\,\mathrm{d}t \ll |c| \ll \frac{T^{\frac{v}{2}}}{\log^2 T}, \tag{3.3.11}$$

since the integrand is $O(1)$ and clearly

$$\int_{W_2} \left(\frac{|t+\beta'|}{2\pi}\right)^{-c}\mathrm{d}t = O(1). \tag{3.3.12}$$

Furthermore, by Lemma 3.3.1, we have

$$\begin{aligned} \int_{W_1} \chi\Big(\frac{1}{2}+a+it\Big)\chi\Big(\frac{1}{2}-b-it\Big)\,\mathrm{d}t = \int_{W_1} e^{-c\log\frac{|t+\beta'|}{2\pi}+(a+it)\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)+c}\,\mathrm{d}t \\ + O(|c|\log T) \end{aligned}$$

and by partial integration

$$\begin{aligned} &\int_{W_1} e^{-c\log\frac{|t+\beta'|}{2\pi}}\left(e^{(a+it)\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)+c}-1\right)\mathrm{d}t =\\ &\qquad= \int_{W_1} \frac{1}{1-c}e^{(1-c)\log\frac{|t+\beta'|}{2\pi}}\left(i\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)+\right.\\ &\qquad\quad \left.+\frac{ic(a+it)}{\left(\frac{1}{2}-a-it\right)\left(\frac{1}{2}-b-it\right)}\right)e^{(a+it)\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)+c}\,\mathrm{d}t+\\ &\qquad\quad + \left[\frac{1}{1-c}e^{(1-c)\log\frac{|t+\beta'|}{2\pi}}\left(e^{(a+it)\log\left(1+\frac{c}{\frac{1}{2}-a-it}\right)+c}-1\right)\right]_{\partial W_1}\\ &\qquad= \int_{W_1} \frac{1}{1-c}e^{(1-c)\log\frac{|t+\beta'|}{2\pi}}\left(O\left(\frac{|c|^2}{|t+\alpha'|^2}\right)\right)\mathrm{d}t+\\ &\qquad\quad + \left[\frac{1}{1-c}e^{(1-c)\log\frac{|t+\beta'|}{2\pi}}\left(O\left(\frac{|c|^2}{|t+\alpha'|}\right)\right)\right]_{\partial W_1}\\ &\qquad\ll |c|\log T \ll \frac{T^{\frac{v}{2}}}{\log T}. \end{aligned}$$

Thus

$$\int_{W_1} \chi\left(\frac{1}{2}+a+it\right)\chi\left(\frac{1}{2}-b-it\right) \mathrm{d}t = \int_{W_1} \left(\frac{|t+\beta'|}{2\pi}\right)^{-c} \mathrm{d}t + O\left(\frac{T^{\frac{v}{2}}}{\log T}\right),$$

and so, by (3.3.10)-(3.3.12), the proof is complete. □

Chapter 4

The first moment of twisted Hecke L-functions

4.1 Introduction

For a positive integer N the Hecke congruence group $\Gamma_0(N)$ is the set of matrices

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

One can define a group action of $\Gamma_0(N)$ on the upper half-plane

$$\mathbb{H} := \{x + iy \,|y > 0; x, y \in \mathbb{R}\},$$

via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z := \frac{az+b}{cz+d},$$

for all $z \in \mathbb{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. The quotient space $\Gamma_0(N)\backslash\mathbb{H}$ is not compact, but one can compactify it by adding finitely many points, called cusps.

One can then define modular forms of level N and weight k as the holomorphic functions defined on $\mathbb{H}$ that are holomorphic at the cusps and satisfy

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. If a modular form f vanishes at all the cusps, then it is called a cusp form. Denoting by $S_k(N)$ the set of cusp forms of level N and weight k, one has that $S_k(N)$ forms a finite dimensional vector space over $\mathbb{C}$, equipped with an inner product, the Petersson inner product $(\,\cdot\,,\cdot\,)$. This is defined by

$$(f, g) := \int_F f(z)\,\overline{g(z)}\,y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2},$$

where $z = x + iy$ and F is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$ (i.e. F is an open subset of $\mathbb{H}$ that intersects every orbit at most once and whose closure intersects every orbit at least once).

Any $f \in S_k(N)$ can be expressed as a Fourier series,

$$f(z) = \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}}\, \mathrm{e}(nz) \qquad \Re(z) > 0,$$

where $\mathrm{e}(z) := e^{2\pi i z}$. The Hecke L-function associated to f is

$$L(s, f) := \sum_{n \geq 1} \frac{a_f(n)}{n^s},$$

where the series is absolutely convergent for $\Re(s) > 1$ by Deligne's bound, $a_f(n) \ll d(n)$, where $d(n)$ is the divisor function. Hecke showed that $L(s, f)$ can be analytically continued to an entire function and satisfies the functional equation

$$\Lambda(s, f) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(2 + \frac{k-1}{2}\right) L(s, f) = i^k \Lambda(1 - s, Wf),$$

where $W : S_k(N) \to S_k(N)$ is the Fricke involution, defined by $Wf(z) := N^{-\frac{k}{2}} z^{-k} f\left(\frac{-1}{Nz}\right)$.

Every cusp form in $S_k(N)$ induces $d(N/M)$ cusp forms in $S_k(M)$ if $N|M$. Thus we can divide $S_k(N)$ into two orthogonal vector spaces

$$S_k(N) = S_k^\flat(N) \oplus S_k^*(N),$$

where $S_k^\flat(N)$ is the subspace of cusp forms induced from lower levels (the "old forms") and $S_k^*(N)$ is its orthogonal complement. Moreover, one has that there exists a unique orthogonal basis of $S^*(N)$ whose elements (the "newforms") satisfy $a_f(1) = 1$ and are eigenfunctions of certain operators (the Hecke operators). We will denote this basis by $\mathcal{H}_k(N)$. Since $S_k(1)$ has dimension 0 if $k \in \{2,4,6,8,10,14\}$, then it follows that $S_k(p) = S_k^*(p)$ for any of these k and for any prime p. In particular, if $k \in \{2,4,6,8,10,14\}$ then $\mathcal{H}_k(p)$ is a basis for $S_k(p)$.

When averaging over any orthonormal basis $\mathcal{F}$ of $S_k(N)$, the coefficients $\{a_f(n)\}_n \cup \{a_{Wf}(m)\}_m$ are approximately orthogonal as the level N goes to infinity, since one has the Petersson formula

$$\begin{aligned}\sideset{}{^{\mathrm{h}}}\sum_{f\in\mathcal{F}} \overline{a_f(m)}\, a_f(n) &= \delta_{m,n} + 2\pi i^{-k} \sum_{\substack{c\geq 1,\\ N\mid c}} \frac{S(m,n;c)}{c} J_{k-1}\bigg(\frac{4\pi\sqrt{mn}}{c}\bigg),\\ \sideset{}{^{\mathrm{h}}}\sum_{f\in\mathcal{F}} \overline{a_{Wf}(m)}\, a_{Wf}(n) &= \delta_{m,n} + 2\pi i^{-k} \sum_{\substack{c\geq 1,\\ N\mid c}} \frac{S(m,n;c)}{c} J_{k-1}\bigg(\frac{4\pi\sqrt{mn}}{c}\bigg),\\ \sideset{}{^{\mathrm{h}}}\sum_{f\in\mathcal{F}} \overline{a_f(m)}\, a_{Wf}(n) &= 2\pi i^{-k} \sum_{\substack{c\geq 1,\\ (c,N)=1}} \frac{S(m\overline{N},n;c)}{c\sqrt{N}} J_{k-1}\bigg(\frac{4\pi\sqrt{mn}}{c\sqrt{N}}\bigg),\end{aligned} \tag{4.1.1}$$

where $\delta_{m,n}$ is the Kronecker's delta function, J_{k-1} is the J-Bessel function,

$$S(m,n,c) := \sum_{\substack{a \pmod{c},\\ (a,c)=1}} \mathrm{e}\bigg(\frac{ma+n\bar{a}}{c}\bigg)$$

is the Kloostermann sum (where $\overline{a}$ is the inverse of a modulo c) and

$$\sideset{}{^{\mathrm{h}}}\sum_{f\in\mathcal{F}} * := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{F}} \frac{*}{(f,f)}$$

is the harmonic average.

Given a cusp form $f \in S_k(N)$ and a primitive Dirichlet character χ modulo q with $(q,N) = 1$ one can define the twisted L-function $L(s, f\otimes\chi)$ via the Dirichlet series

$$L(s,f\otimes\chi) := \sum_{n\geq 1} \frac{\chi(n)\, a_f(n)}{n^s},$$

which is absolutely convergent for $\Re(s) > 1$ and can be analytically continued to an entire function. Moreover, $L(s, f \otimes \chi)$ satisfies the functional equation

$$\begin{aligned}\Lambda(s, f \otimes \chi) &:= \left(\frac{\sqrt{N}q}{2\pi}\right)^{s} \Gamma\left(s + \frac{k-1}{2}\right) L(s, f \otimes \chi) \\ &= i^k \omega(\chi) \Lambda(1-s, Wf \otimes \overline{\chi}),\end{aligned} \tag{4.1.2}$$

where $\omega(\chi) := \chi(N)\tau(\chi)^2/N$ and $\tau(\chi) := \sum_{a=1}^{q} \chi(a)\, \mathrm{e}\left(n\frac{a}{q}\right)$ is the Gauss sum. (For this and for the aforementioned properties of Hecke L-functions see, for example, [Iwa97])

L-functions associated to modular forms have been studied extensively with applications in many directions of number theory. In this chapter we focus on averages of Hecke L-functions twisted by a primitive Dirichlet character χ of conductor coprime with the level. The (twisted) L-functions associated to newforms of a given weight form an orthogonal family in the sense of Katz and Sarnak [KaS99a], thus, for a primitive Dirichlet character χ with conductor q coprime with N, one expects that

$$\sum_{f \in \mathcal{H}_k(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f \otimes \chi\right)^{m} = P_{m,k,\chi}(\log N) + o(1), \tag{4.1.3}$$

where $P_{m,k,\chi}$ is a polynomial of degree $\frac{m(m-1)}{2}$ and the implied constant may depend on q and k.

Duke [Duk] computed the asymptotics (4.1.3) for the first and the second moment (i.e. $m = 1, 2$), provided that N is prime and $k = 2$, with an error term of size $O\left(N^{-\frac{1}{2}+\varepsilon}\right)$. For the first moment, Ellenberg [Ell] improved the bound for the error term to $O(N^{-1+\varepsilon})$. He needed this better estimate to tackle the problem of finding all primitive solutions to the generalized Fermat equation $a^2 + b^2 = c^p$.

In the pioneering work [IS], Iwaniec and Sarnak studied the first and second moment (both in the level and the weight aspects) in the case of real characters. They showed that for $m = 1, 2$ the asymptotics (4.1.3) holds for all even $k \geq 2$ and they relaxed also the condition on the primality of N, replacing it by

$\frac{\varphi(N)}{N} \to 1$, where $\varphi(n)$ is Euler's totient function. They studied this asymptotic in an attempt to show that there are no Siegel zeros. These are hypothetical real zeros of (quadratic) Dirichlet L-functions $L\big(\frac{1}{2}, \chi\big)$ which are "close" to 1. Iwaniec and Sarnak showed that the non-existence of such exceptional zeros would follow if one could prove the non-vanishing (with some additional lower bound) of strictly more than $\frac{1}{4}$ of the central values of the Hecke L-functions (asymptotically when either the level or the weight goes to infinity). They approached the problem by considering mollified first and second moment for Hecke L-functions twisted by real characters, but unfortunately they were able to reach but not surpass the limit of $\frac{1}{4}$.

The asymptotics for the (mollified) fourth moment was proved by Kowalski, Michel and VanderKam [KMV] for prime levels. From this result they also deduced the non-vanishing of a positive proportion of the central values of $L(s,f)L(s,f\otimes\chi)$ for non-real characters. (For other applications of results on moments of Hecke L-functions see, among others, [DFI], [KM] and [Van].)

It is often useful to consider the shifted moments,

$$\mathcal{M}_k(\alpha_1, \dots \alpha_m, \chi; N) := \sum_{f\in\mathcal{H}_k(N)}^{\mathrm{h}} L\bigg(\frac{1}{2}+\alpha_1, f\otimes\chi\bigg)\cdots L\bigg(\frac{1}{2}+\alpha_m, f\otimes\chi\bigg),$$

as these reveal more clearly the combinatorics behind the main terms. Usually the shifts are taken to be fixed (or less than q^{ε} for some small $\varepsilon > 0$), however when studying the n-correlation of zeros one would like to apply conjectures on moments of ratios of shifted L-functions and integrate over the shifts. Thus, one needs to understand for what range of shifted parameters the asymptotics for the moments still holds.

In this chapter we shall consider the shifted first moment. Kamiya addressed this problem in [Kam], showing that if N is prime and $\Re(\alpha) = 0$ then

$$\mathcal{M}_k(\alpha, \chi; N) \sim 1 \tag{4.1.4}$$

for $k \in \{2,4,6,8,10,14\}$ and $qT \ll N^{\frac{1}{2}-\varepsilon}$, where $T := 1 + |\Im(\alpha)|$. The following theorem extends the range of validity of (4.1.4) to $qT \ll N^{1-\varepsilon}$. We take $k = 2$ for simplicity, however the result should be easily generalizable to all $k \in \{2, 4, 6, 8, 10, 14\}$.

Theorem 4.1.1. *Let N be prime and let χ be a primitive character modulo q with $(q, N) = 1$. Let $|\Re(\alpha)| \ll \frac{1}{\log N}$ and write $T = 1 + |\Im(\alpha)|$. Then, if $Tq < N^{1-\varepsilon}$, for all $\varepsilon > 0$ one has*

$$\mathcal{M}_2(\alpha, \chi; N) = 1 + O\left(\frac{\varphi(q)\sqrt{T}}{N^{1-\varepsilon}}\right), \tag{4.1.5}$$

as N goes to infinity.

The main idea of the proof comes essentially from Ellenberg [Ell], however there are two complications that are not present in the unshifted case. In both proofs the first step is to use an approximate functional equation to express $L\big(\frac{1}{2}, f \otimes \chi\big)$ as a sum of two Dirichlet series of length roughly $\sqrt{N}TqY$ and $\sqrt{N}Tq/Y$ for a parameter Y, where the second sum is multiplied by the sign of the functional equation. If $Y = 1$, after averaging over $\mathcal{H}_k(N)$ the second sum contributes more than the first. This is due to the worse control that we have on the orthogonality of $a_f(n)$ with $a_{Wf}(m)$ (compare the first and the second equation with the third one in (4.1.1)). This leads to an optimal choice of $Y = \sqrt{N}$ and, in particular, if qT is fixed one can avoid having to deal with the second sum. However, if qT is large one needs to find a good upper bound also for the second sum.

The second difference is that in the unshifted case one can use the exponential function in the approximate functional equation, that is one uses

$$L\left(\frac{1}{2}, f \otimes \chi\right) \approx \sum_{n \geq 1} \frac{\chi(n)\, a_f(n)}{n^{\frac{1}{2}}}\, e^{-\frac{n}{N^{1+\varepsilon}}}$$

and later exploits the additivity of e^x. In our case, we have another function in place of the exponential and we replace the additivity by using some basic properties of the twisted periodic zeta functions. This approach is similar to that of [BH], where Blomer and Harcos obtained hybrid bounds for the second moment of twisted Hecke L-functions.

Finally, we make a couple of remarks on the error term in (4.1.5). First we observe that the bound $O\Big(\frac{\varphi(q)\sqrt{T}}{N^{1-\varepsilon}}\Big)$ suggests T could be as large as $N^{2-\varepsilon}$ if q is fixed. However, the condition $Tq < N^{1-\varepsilon}$ is used crucially to obtain that bound. It appears to us that in the range $qT > N^{1+\varepsilon}$ the sum over $c \le \sqrt{mn}$ in the Petersson formula (4.1.1) is no longer negligible and perhaps one should try to obtain some extra savings by deploying the oscillation in the J-Bessel function.

Second, we observe that the $\varphi(q)$ in the error term (4.1.5) comes from bounding trivially the sum

$$\sum_{\ell=1}^{q} \chi(\ell)\,\mathrm{e}\Big(\ell\frac{a}{b}\Big),$$

where b is allowed to be coprime with the conductor q of χ. If one could obtain a better bound with respect to q, then this would go straight into the error term (4.1.5) (although one would still need to assume $Tq < N^{1-\varepsilon}$).

4.2 Preliminaries

Remark 4.2.1. *From now on we will write* $T := |\Im(\alpha)| + 1$ *and assume that* $T, q \ll N^{100}$ *and*

$$\Re(\alpha) \ll \frac{1}{\log N}. \tag{4.2.1}$$

We shall compute the asymptotics of

$$M_2(\alpha,\chi;N) = \sum_{f\in\mathcal{F}}^{\mathrm{h}} \overline{a_f(1)} L\Big(\frac{1}{2}+\alpha, f\otimes\chi\Big),$$

for any orthonormal basis $\mathcal{F}$ of $S_2(N)$, where N is not necessarily prime. If N is prime, then taking $\mathcal{F}$ to be the orthonormal basis of $S_2(N)$ formed by all the functions in $\mathcal{H}_2(N)$ divided by their norms, one has that

$$M_2(\alpha,\chi;N) = \mathcal{M}_2(\alpha,\chi;N).$$

Next, we express $L\big(\frac{1}{2}+\alpha, f\otimes\chi\big)$ as an (almost) finite sum, by using an approximate functional equation.

Lemma 4.2.2. *Let $f \in S_k(N)$ and let χ be a primitive Dirichlet character modulo q with $(q,N)=1$. Let $G(s)$ be an even function which is bounded on vertical strips and is such that $G(0)=1$. Then, for $Y>0$ we have*

$$\begin{aligned} L\left(\frac{1}{2}+\alpha, f\right) = \sum_{n\geq 1} \frac{\chi(n)a_f(n)}{n^{\frac{1}{2}+\alpha}} V_\alpha\left(\frac{n}{q\sqrt{N}Y}\right) + \\ - \omega(\chi)X_\alpha \sum_{n\geq 1}\frac{\overline{\chi}(n)a_{Wf}(n)}{n^{\frac{1}{2}-\alpha}} V_{-\alpha}\left(\frac{nY}{q\sqrt{N}}\right), \end{aligned} \tag{4.2.2}$$

where

$$\begin{aligned} V_\alpha(x) &:= \frac{1}{2\pi i}\int_{(2)} G(s)(2\pi x)^{-s}\frac{\Gamma(1+\alpha+s)}{\Gamma(1+\alpha)}\frac{\mathrm{d}s}{s}, \\ X_\alpha &:= \left(\frac{q\sqrt{N}}{2\pi}\right)^{-2\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}. \end{aligned} \tag{4.2.3}$$

Proof. Exchanging the order of summation and integration, the first sum on the right hand side of (4.2.2) is equal to $\Gamma(1+\alpha)\left(\frac{\sqrt{N}q}{2\pi}\right)^{\frac{1}{2}+\alpha}$ times

$$\frac{1}{2\pi i}\int_{(2)} G(s)\Lambda\left(\frac{1}{2}+s+\alpha, f\otimes\chi\right)\frac{\mathrm{d}s}{s}.$$

Moving the line of integration to $\Re(s) = -2$ and applying the functional equation (4.1.2) we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{(2)} G(s)\Lambda\left(\frac{1}{2}+s+\alpha, f\otimes\chi\right)\frac{\mathrm{d}s}{s} = \\ = \Lambda\left(\frac{1}{2}+\alpha, f\otimes\chi\right) - \omega(\chi)\frac{1}{2\pi i}\int_{(-2)} G(s)\Lambda\left(\frac{1}{2}-s-\alpha, Wf\otimes\chi\right)\frac{\mathrm{d}s}{s} \end{aligned}$$

and the Lemma follows after the change of variable $s \to -s$. □

Remark 4.2.3. *In order to avoid exponential growth of $V_\alpha(x)$ with respect to $T := |\Im(\alpha)|+1$ we need to take a function $G(s)$ that decays fast on vertical strips. Moreover, we shall assume that $G(s)$ has zeros at $s = 1-\alpha+\ell$ for all integers $0 \leq \ell \leq L$, where L is a fixed and large positive number. To make this explicit we take*

$$G(s) := e^{s^2}\prod_{\ell=0}^{L}\left(s^2-(1-\alpha+\ell)^2\right).$$

With this choice of $G(s)$*, by Stirling's formula we have*

$$\begin{aligned} G(s)\frac{\Gamma(1+\alpha+s)}{\Gamma(1+\alpha)} &\ll T^{\sigma}e^{-\frac{\pi}{2}(|s+\alpha|-|\alpha|)}|G(s)| \\ &\ll T^{\sigma}e^{-\frac{|s|^2}{2}}, \end{aligned} \tag{4.2.4}$$

for $s+\alpha-m \gg 1$ *for all non-negative integer* m *(and* α *satisfying* (4.2.1)*). Thus, moving the line of integration in* (4.2.3) *to the left and to the right, we have*

$$V_{\alpha}(x) \ll \min\bigl(1, T^{B}x^{-B}\bigr), \tag{4.2.5}$$

for all fixed $B > 0$*. This implies that the two Dirichlet series in* (4.2.2) *have approximately* $q\sqrt{N}TY$ *and* $q\sqrt{N}T/Y$ *significant terms.*

Remark 4.2.4. *Throughout the rest of the chapter we will assume* $1 \leq Y \ll X^{100}$*. We will choose* Y *optimally later.*

Using the approximate functional equation (4.2.2), we can decompose $M_2(\alpha,\chi;N)$ into

$$M_2(\alpha,\chi;N) = B_1(\alpha,\chi) - \omega(\chi)X_{\alpha}B_{-1}(\alpha,\chi), \tag{4.2.6}$$

where

$$\begin{aligned} B_1(\alpha,\chi) &:= \sum_{n\geq 1}\frac{\chi(n)}{n^{\frac{1}{2}+\alpha}}\Biggl(\sideset{}{^{\mathrm{h}}}\sum_{f\in F}\overline{a_f(1)}\,a_f(n)\Biggr)V_{\alpha}\biggl(\frac{n}{q\sqrt{N}Y}\biggr), \\ B_{-1}(\alpha,\chi) &:= \sum_{n\geq 1}\frac{\overline{\chi}(n)}{n^{\frac{1}{2}-\alpha}}\Biggl(\sideset{}{^{\mathrm{h}}}\sum_{f\in F}\overline{a_f(1)}\,a_{Wf}(n)\Biggr)V_{-\alpha}\biggl(\frac{nY}{q\sqrt{N}}\biggr). \end{aligned}$$

We can now apply the Petersson formula (4.1.1) getting

$$B_1(\alpha,\chi) = V_{\alpha}\biggl(\frac{1}{q\sqrt{N}Y}\biggr) + E(\alpha,\chi), \tag{4.2.7}$$

where

$$E(\alpha,\chi) := -2\pi\sum_{n\geq 1}\frac{\chi(n)}{n^{\frac{1}{2}+\alpha}}\sum_{\substack{c\geq 1,\\ N|c}}\frac{S(1,n;c)}{c}J_1\biggl(\frac{4\pi\sqrt{n}}{c}\biggr)V_{\alpha}\biggl(\frac{n}{q\sqrt{N}Y}\biggr), \tag{4.2.8}$$

and

$$B_{-1}(\alpha,\chi) = -2\pi \sum_{n\geq 1} \frac{\overline{\chi}(n)}{n^{\frac{1}{2}-\alpha}} \sum_{\substack{c\geq 1,\\ (c,N)=1}} \frac{S(\overline{N},n;c)}{c\sqrt{N}} J_1\bigg(\frac{4\pi\sqrt{n}}{c\sqrt{N}}\bigg) V_{-\alpha}\bigg(\frac{nY}{q\sqrt{N}}\bigg). \tag{4.2.9}$$

Next, we remove large values of c. Using the inequalities (4.2.5) and $J_1(x) \ll x$ and Weyl's bound

$$|S(m,n;c)| \leq d(c)(m,n,c)^{\frac{1}{2}} c^{\frac{1}{2}},$$

where $d(c)$ is the divisor function, we have that the contribution to $E(\alpha,\chi)$ coming from the c that are larger than C is bounded by a constant times

$$\frac{1}{N^{\frac{3}{2}}} \sum_{c>C} \frac{d(c)}{c^{\frac{3}{2}}} \sum_{n\geq 1} \frac{1}{n^{\Re(\alpha)}} \bigg| V_\alpha\bigg(\frac{n}{q\sqrt{N}Y}\bigg)\bigg| \ll \frac{qTY}{(N\sqrt{C})^{1-\frac{1}{100}}},$$

which is $O(N^{-2})$ if we take $C = N^D$ with D large enough. Clearly we can do the same with $B_{-1}(\alpha,\chi)$ and thus we truncate both sums at height C, where from now on C will be equal to N^D for some fixed and large D.

Thus, opening the Kloosterman sum and exchanging the order of summation, as allowed by (4.2.5), it follows that

$$\begin{aligned} E(\alpha,\chi) &= -2\pi \sum_{\substack{c\leq C,\\ N|c}} \frac{1}{c} \sum_{\substack{a \pmod{c_1},\\ (a,c_1)=1}} e\bigg(\frac{\overline{a}}{c_1}\bigg) T_1(a,c,\alpha,\chi) + O\big(N^{-2}\big), \\ B_{-1}(\alpha,\chi) &= -2\pi \sum_{\substack{c_{-1}\leq C,\\ (c_{-1},N)=1}} \frac{1}{c} \sum_{\substack{a \pmod{c_{-1}},\\ (a,c_{-1})=1}} e\bigg(\frac{\overline{aN}}{c_{-1}}\bigg) T_{-1}(a,c,-\alpha,\overline{\chi}) + O\big(N^{-2}\big), \end{aligned} \tag{4.2.10}$$

where we write $c_1 = c$, $c_{-1} = cN^{-\frac{1}{2}}$ and for $\epsilon = \pm 1$

$$T_\epsilon(a,c,\alpha,\chi) := \sum_{n\geq 1} \frac{\chi(n)\, \mathrm{e}\Big(\frac{na}{c_\epsilon}\Big)}{n^{\frac{1}{2}+\alpha}} J_1\bigg(\frac{4\pi\sqrt{n}}{c}\bigg) V_\alpha\bigg(\frac{n}{q\sqrt{N}Y^\epsilon}\bigg). \tag{4.2.11}$$

4.3 The Lerch zeta function and the twisted periodic zeta function

In order to bound $T_\epsilon(a,c,\alpha,\chi)$ we will need some properties of the twisted periodic zeta function,

$$F\Big(s,\chi,\frac{a}{c}\Big) := \sum_{n\geq 1}\frac{\chi(n)\,\mathrm{e}\big(\frac{na}{c}\big)}{n^s}, \tag{4.3.1}$$

for $\Re(s) > 1$ and where $(a,c)=1$ and χ is a primitive Dirichlet character modulo q. Dividing the sum over n according to the class of n modulo q, we can decompose $F\big(s,\chi,\frac{a}{c}\big)$ into a linear combination of Lerch's zeta functions,

$$F\Big(s,\chi,\frac{a}{c}\Big) = \frac{1}{q^s}\sum_{\ell=1}^{q-1}\chi(\ell)\,\mathrm{e}\Big(\frac{\ell a}{c}\Big)\mathcal{L}\Big(s,\frac{qa}{c},\frac{\ell}{q}\Big), \tag{4.3.2}$$

where for $\Re(s)>1$ the Lerch zeta function is defined by

$$\mathcal{L}(s,x,y) := \sum_{n\geq\{1-y\}}\frac{\mathrm{e}(n\{x\})}{(n+\{y\})^s}, \tag{4.3.3}$$

where $\{x\}$ is the fractional part of x. (Notice that we have "periodized" the usual definition of the Lerch zeta function as this will simplify the notation later.)

The following lemma gives the analytic continuation and the functional equation for the Lerch zeta function.

Lemma 4.3.1. *Let $x,y\in\mathbb{R}$. We have that $\mathcal{L}(s,x,y)$ can be analytically continued as a function of s to a holomorphic function in $\mathbb{C}$ with the exception of a simple pole at $s=1$ of residue 1 if $x\in\mathbb{Z}$. Moreover, $\mathcal{L}(s,x,y)$ satisfies the functional equation*

$$\begin{aligned}\mathcal{L}(1-s,x,y) = (2\pi)^{-s}\Gamma(s)\Big(&e^{\frac{\pi i s}{2}-2\pi i\{y\}\{x\}}\mathcal{L}(s,-y,x)+\\ &+e^{-\frac{\pi i s}{2}+2\pi i\{y\}\{-x\}}\mathcal{L}(s,y,-x)\Big).\end{aligned} \tag{4.3.4}$$

Proof. See, for example, [Ler]. □

By (4.3.2) we can deduce the following corollary.

Corollary 4.3.2. *Let $(a, c) = 1$ and let χ be a primitive Dirichlet character modulo q. Then $F\big(s, \chi, \frac{a}{c}\big)$ is an entire function of s with the exception of a simple pole at $s = 1$ of residue*

$$\frac{1}{q^s}\sum_{\ell=1}^{q-1}\chi(\ell)\,\mathrm{e}\bigg(\frac{\ell a}{c}\bigg)$$

if $c|q$.

We need to be able to control the dependence of $F\big(s, \chi, \frac{a}{c}\big)$ on $\frac{a}{c}$ when $\Re(s)$ is close to 0. If $\Re(s) > 1+\varepsilon$ and $y \notin \mathbb{Z}$, it is easy to see that $\mathcal{L}(s, x, y)$ is bounded once we have removed the term $n = 0$ from the sum (4.3.3), i.e. we have

$$\mathcal{L}(s, x, y) - \{y\}^{-s} \ll 1. \tag{4.3.5}$$

By the functional equation this argument can be extended to $\Re(s)$ close to 0.

Lemma 4.3.3. *Let $y \notin \mathbb{Z}$ and let*

$$\begin{aligned}\mathcal{L}_*(s, x, y) := i(2\pi)^{s-1}\Gamma(1-s)\big(&e^{-\frac{\pi i s}{2}-2\pi i\{y\}\{x\}}\{x\}^{s-1}+\\ &-e^{\frac{\pi i s}{2}+2\pi i\{y\}\{-x\}}\{-x\}^{s-1}\big),\end{aligned} \tag{4.3.6}$$

if $x \notin \mathbb{Z}$ and let $L_*(s, x, y) = 0$ if $x \in \mathbb{Z}$. Moreover, let

$$\mathcal{L}^*(s, x, y) := \mathcal{L}(s, x, y) - \mathcal{L}_*(s, x, y).$$

Then if $-B \le \Re(s) \le \varepsilon$, where $\varepsilon > 0$ and $B > \varepsilon$ are fixed, we have

$$\mathcal{L}^*(s, x, y) = O\Big((|\Im(s)| + 1)^{\frac{1}{2}+B}\{y\}^{-\varepsilon}\Big), \tag{4.3.7}$$

uniformly in x and y.

Proof. By the functional equation (4.3.4) and (4.3.5) it follows that for $-B \le \Re(s) \le -\varepsilon$ one has

$$\mathcal{L}^*(s, x, y) = O\Big((|\Im(s)| + 1)^{\frac{1}{2}+B}\Big).$$

Moreover, by (4.3.5) on the line $\Re(s) = 1+\varepsilon$ we have that

$$\mathcal{L}^*(s,x,y) \ll \{y\}^{-1-\varepsilon}.$$

The Lemma then follows by applying the Phragmén-Lindelöf theorem. □

From (4.3.2) we can then easily deduce the analogous result for $F\big(s,\chi,\frac{a}{c}\big)$.

Corollary 4.3.4. *Let χ be a primitive Dirichlet character modulo q and let $(a,c)=1$. Let*

$$\begin{aligned}F_*\Big(s,\chi,\frac{a}{c}\Big) &:= \psi(\chi,a,c)\frac{\Gamma(1-s)}{(2\pi)^{1-s}q^s}\times\\ &\times\bigg(e^{-\frac{\pi is}{2}}\Big\{\frac{qa}{c}\Big\}^{s-1} - \mathrm{e}\Big(\frac{qa}{c}\Big)e^{\frac{\pi is}{2}}\Big\{-\frac{qa}{c}\Big\}^{s-1}\bigg),\end{aligned} \tag{4.3.8}$$

if $c\nmid q$, where

$$\psi(\chi,a,c) := i\sideset{}{^*}\sum_{\ell \ (\mathrm{mod}\ q)}\chi(\ell)\,\mathrm{e}\bigg(\frac{\ell a}{c} - \Big\{\frac{qa}{c}\Big\}\frac{\ell}{q}\bigg),$$

and let $F^*\big(s,\chi,\frac{a}{c}\big) := 0$ if $c \mid q$. Moreover, let

$$F^*\Big(s,\chi,\frac{a}{c}\Big) := F\Big(s,\chi,\frac{a}{c}\Big) - F_*\Big(s,\chi,\frac{a}{c}\Big). \tag{4.3.9}$$

Then, if $-B \le \Re(s) \le \varepsilon$, where $\varepsilon > 0$ and $B > \varepsilon$ are fixed, we have

$$F^*\Big(s,\chi,\frac{a}{c}\Big) \ll \varphi(q)\,q^{B+\varepsilon}\Big(1+|\Im(s)|^{\frac{1}{2}+B}\Big), \tag{4.3.10}$$

where the implied constant depends only on ε and B.

4.4 Bounding $T_\epsilon(c,\alpha,\chi)$

We start by recalling the Mellin transform of $J_1(x)$,

$$J_1(x) = \frac{1}{2\pi i}\int_{(-\delta)} 2^{s-1}\frac{\Gamma\big(\frac{s+1}{2}\big)}{\Gamma\big(\frac{3}{2}-\frac{s}{2}\big)}x^{-s}\,\mathrm{d}s,$$

for any $0<\delta<1$. Thus, from (4.2.11) we have

$$T_\epsilon(a,c,\alpha,\chi) = \frac{1}{4\pi i}\int_{(-\delta)}\frac{\Gamma\big(\frac{s+1}{2}\big)}{\Gamma\big(\frac{3}{2}-\frac{s}{2}\big)}\Big(\frac{2\pi}{c}\Big)^{-s}\sum_{n\ge1}\frac{\chi(n)\,\mathrm{e}\Big(\frac{na}{c_\epsilon}\Big)}{n^{\frac{1}{2}+\alpha+\frac{s}{2}}}V_\alpha\Big(\frac{n}{q\sqrt{N}Y^\epsilon}\Big)\mathrm{d}s,$$

for $\epsilon = \pm 1$. Now, using the integral representation (4.2.3) of V_α, we have

$$\begin{aligned} T_\epsilon(a, c, \alpha, \chi) =& \frac{1}{2(2\pi i)^2} \int_{(-\delta)} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{s}{2}\right)} \left(\frac{2\pi}{c}\right)^{-s} \int_{(\gamma)} \frac{G(w)\Gamma(1+\alpha+w)}{\Gamma(1+\alpha)} \times \\ & \times F\left(\frac{1}{2} + \alpha + \frac{s}{2} + w, \chi, \frac{a}{c_\epsilon}\right) \left(\frac{q\sqrt{N}Y^\epsilon}{2\pi}\right)^w \frac{\mathrm{d}w}{w}\,\mathrm{d}s, \end{aligned} \tag{4.4.1}$$

for $\Re(s) = \delta$ and $\gamma = \frac{1}{2} - \Re(\alpha) + \frac{\delta}{2} + \varepsilon$. We make the change of variable $w \to w - \frac{s}{2}$, so that, after changing the order of integration, as is clearly possible by absolute convergence of the integrals, $T_\epsilon(a, c, \alpha, \chi)$ becomes

$$\begin{aligned} T_\epsilon(a, c, \alpha, \chi) = \frac{1}{2\pi i} \int_{\left(\gamma - \frac{\delta}{2}\right)} & F\left(\frac{1}{2} + \alpha + w, \chi, \frac{a}{c_\epsilon}\right) \times \\ & \times \left(\frac{q\sqrt{N}Y}{2\pi}\right)^w W\left(w, \frac{2\pi q\sqrt{N}Y}{c^2}\right) \mathrm{d}w, \end{aligned} \tag{4.4.2}$$

where

$$W(w, x) := \frac{1}{4\pi i} \int_{(-\delta)} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{s}{2}\right)} \frac{G\left(w - \frac{s}{2}\right)\Gamma\left(1 + \alpha + w - \frac{s}{2}\right)}{\Gamma(1+\alpha)} x^{-\frac{s}{2}} \frac{\mathrm{d}s}{w - \frac{s}{2}}.$$

Assuming $\Re(w) = \gamma - \frac{\delta}{2} = \frac{1}{2} - \Re(\alpha) + \varepsilon$, we move the line of integration to $\Re(s) = -3 - 2M + 2\varepsilon$ for some integer $M \geq 0$, encountering simple poles at the odd negative integers. Thus we obtain the asymptotic expansion

$$\begin{aligned} W(w, x) = & \sum_{m=0}^{M} \frac{(-1)^m x^{\frac{1}{2}+m}}{m!(m+1)!} \frac{G\left(\frac{1}{2} + w + m\right)\Gamma\left(\frac{3}{2} + \alpha + w + m\right)}{\left(\frac{1}{2} + w + m\right)\Gamma(1+\alpha)} + \\ & + O\left(\frac{T^{\frac{1}{2}}(Tx)^{\frac{3}{2}+M-\varepsilon}}{(|w|+1)^{4+2M-\varepsilon}}\right), \end{aligned} \tag{4.4.3}$$

since by (4.2.4) and Stirling's formula we have

$$\int_{(-3-2M+\varepsilon)} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{s}{2}\right)} \frac{G\left(w - \frac{s}{2}\right)\Gamma\left(1 + \alpha + w - \frac{s}{2}\right)}{\left(w - \frac{s}{2}\right)\Gamma(1+\alpha)} x^{-\frac{s}{2}} \mathrm{d}s \ll \frac{T^{\frac{1}{2}}(Tx)^{\frac{3}{2}+M-\varepsilon}}{(|w|+1)^{4+2M-\varepsilon}},$$

with an implied constant that may depend on M (and ε).

Therefore, inserting the expansion (4.4.3) in (4.4.2) we have

$$T_\epsilon(a, c, \alpha, \chi) = \frac{2\pi}{c} \sum_{m=0}^{M} \frac{(-1)^m \left(\frac{2\pi q\sqrt{N}Y^\epsilon}{c^2}\right)^m}{m!(m+1)!} H_{\epsilon,\alpha,m}(a, c, \alpha, \chi) + \mathcal{E}_{\epsilon,\alpha}(a, c, \alpha, \chi), \tag{4.4.4}$$

where

$$H_{\epsilon,\alpha,m}(a,c,\alpha,\chi) := \frac{1}{2\pi i}\int_{\left(\gamma-\frac{\delta}{2}+\frac{1}{2}\right)} \frac{G(w+m)\Gamma(1+\alpha+w+m)}{(w+m)\Gamma(1+\alpha)}\times \\ \times F\left(\alpha+w,\chi,\frac{a}{c_\epsilon}\right)\left(\frac{q\sqrt{N}Y^\epsilon}{2\pi}\right)^w \mathrm{d}w$$

and

$$\mathcal{E}_{\epsilon,\alpha}(a,c) \ll T^{\frac{1}{2}}\left(\frac{2\pi q\sqrt{N}Y^\epsilon T}{c^2}\right)^{\frac{3}{2}+M-\varepsilon}, \tag{4.4.5}$$

since by the definition (4.3.1), for $\Re(s) > 1+\varepsilon$ one has $F\big(s,\chi,\frac{a}{c}\big) \ll 1$.

Now, use (4.3.9) to break $H_{\epsilon,\alpha,m}(a,c,\alpha,\chi)$ into

$$H_{\epsilon,\alpha,m}(a,c,\alpha,\chi) = H^*_{\epsilon,\alpha,m}(a,c,\alpha,\chi) + H_{*,\epsilon,\alpha,m}(a,c,\alpha,\chi)$$

in the way suggested by the notation. Firstly we consider

$$H^*_{\epsilon,\alpha,m}(a,c,\alpha,\chi) := \frac{1}{2\pi i}\int_{\left(\gamma-\frac{\delta}{2}+\frac{1}{2}\right)} \frac{G(w+m)\Gamma(1+\alpha+w+m)}{(w+m)\Gamma(1+\alpha)}\times \\ \times F^*\left(\alpha+w,\chi,\frac{a}{c_\epsilon}\right)\left(\frac{q\sqrt{N}Y^\epsilon}{2\pi}\right)^w \mathrm{d}w$$

and we observe that, by Corollary (4.3.2) and Remark 4.2.3 (assuming that L is large enough compared to M), we can move the line of integration to $\Re(w) = -m+\varepsilon$ without encountering any pole. We bound the integral on the new line and, by (4.2.4) and (4.3.10), we get

$$H^*_{\epsilon,\alpha,m}(a,c,\alpha,\chi) \ll \left(\sqrt{N}Y^\epsilon\right)^{-m+\varepsilon} T^{m-\varepsilon+\frac{1}{2}}\varphi(q)\,q^\varepsilon. \tag{4.4.6}$$

Next, we consider

$$H_{*,\epsilon,\alpha,m}(a,c,\alpha,\chi) := \frac{1}{2\pi i}\int_{\left(\gamma-\frac{\delta}{2}+\frac{1}{2}\right)} \frac{G(w+m)\Gamma(1+\alpha+w+m)}{(w+m)\Gamma(1+\alpha)}\times \\ \times F_*\left(\alpha+w,\chi,\frac{a}{c_\epsilon}\right)\left(\frac{q\sqrt{N}Y^\epsilon}{2\pi}\right)^w \mathrm{d}w.$$

We can assume $c_\epsilon \nmid q$, since by definition $F_*\Big(\alpha+w,\chi,\frac{a}{c_\epsilon}\Big) \equiv 0$ if $c_\epsilon | q$. We move the line of integration to $\Re(w) = \varepsilon$. From the definition (4.3.8), for $\Re(w) = \varepsilon$

we have

$$F_*\left(\alpha+w,\chi,\frac{a}{c_\epsilon}\right)\ll\varphi(q)\,T^{\frac{1}{2}+\varepsilon}\left(\left\{\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1}+\left\{-\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1}\right),$$

by the trivial bound $|\psi_{a,c,q}|\ll\varphi(q)$. Thus, by (4.3.2) and (4.2.4) we have

$$H_{*,\epsilon,\alpha,m}(a,c,\alpha,\chi)\ll N^\varepsilon\varphi(q)\,T^{\frac{1}{2}+m}\left(\left\{\frac{qa}{c}\right\}^{\varepsilon-1}+\left\{-\frac{qa}{c}\right\}^{\varepsilon-1}\right).\tag{4.4.7}$$

Thus, by (4.4.4)-(4.4.7), for $\epsilon=\pm1$ we have

$$\begin{aligned}T_\epsilon(a,c,\alpha,\chi)\ll{}&T^{\frac{1}{2}}N^\varepsilon\left(\frac{q\sqrt{N}Y^\epsilon T}{c^2}\right)^{\frac{3}{2}+M-\varepsilon}+\frac{T^{\frac{1}{2}}\varphi(q)\,N^\varepsilon}{c}\sum_{m=0}^{M}\left(\frac{qT}{c^2}\right)^m+\\&+\frac{T^{\frac{1}{2}}\varphi(q)\,N^\varepsilon}{c}\sum_{m=0}^{M}\left(\frac{q\sqrt{N}TY^\epsilon}{c^2}\right)^m\left(\left\{\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1}+\left\{-\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1}\right),\end{aligned}\tag{4.4.8}$$

where the second line has to be omitted if $c_\epsilon|q$.

4.5 Proof of Theorem 4.1.1

We can now use (4.4.8) to bound the two sums in (4.2.10). Recalling that in $E(\alpha,\chi)$ we are summing over c that are multiples of N and in $B_{-1}(\alpha,\chi)$ we are summing over "multiples of$\sqrt{N}$", it is clear that the optimal choice for Y in (4.4.8) is to take $Y=\sqrt{N}$ and thus we make this choice.

Now, we assume that $qT<N$ (if $qT>N$ we might be tempted to take $M=-1+2\varepsilon$ in (4.4.8), as it would have been possible, however this would lead at the end to a final error of size $\frac{Tq^{\frac{1}{2}}}{N^{\frac{1}{2}-\varepsilon}}$, which is too large). With this assumption (4.4.8) becomes

$$\begin{aligned}T_\epsilon(a,c,\alpha,\chi)\ll{}&T^{\frac{1}{2}}N^\varepsilon\left(\frac{qTN^{\frac{1}{2}+\frac{\epsilon}{2}}}{c^2}\right)^{\frac{3}{2}+M-\varepsilon}+\\&+\frac{T^{\frac{1}{2}}\varphi(q)\,N^\varepsilon}{c}\left(1+\left\{\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1}+\left\{-\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1}\right),\end{aligned}\tag{4.5.1}$$

where the second line has to be omitted if $c_\epsilon | q$. Also, for $\epsilon = \pm 1$ and any integers b and q with $(c_\epsilon, q) = 1$ we have

$$\begin{aligned}
\sum_{\substack{a \pmod{c_\epsilon},\\ (a,c_\epsilon)=1}} e\left(\frac{\overline{a}b}{c_\epsilon}\right)\left(\left\{\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1} + \left\{-\frac{qa}{c_\epsilon}\right\}^{\varepsilon-1}\right) &= \\
= \sum_{\substack{\pmod{c_\epsilon},\\ (a,c_\epsilon)=1}}^{*} e\left(\frac{\overline{a}bq}{c_\epsilon}\right)\left(\left\{\frac{a}{c_\epsilon}\right\}^{\varepsilon-1} + \left\{-\frac{a}{c_\epsilon}\right\}^{\varepsilon-1}\right) \qquad (4.5.2) \\
\ll \sum_{a=1}^{c_\epsilon}\left(\frac{c_\epsilon}{a}\right)^{1-\varepsilon} \ll c_\epsilon
\end{aligned}$$

and, collecting the common factors, the same clearly holds for all c_ϵ not dividing q. Thus, using (4.5.1) and (4.5.2) in (4.2.10), we have

$$\begin{aligned}
E(\alpha, \chi) &\ll N^D T^{\frac{1}{2}} N^{\varepsilon}\left(\frac{qT}{N}\right)^{\frac{3}{2}+M-\varepsilon} + \frac{T^{\frac{1}{2}}\varphi(q)}{N^{1-\varepsilon}}, \\
B_{-1}(\alpha, \chi) &\ll N^D T^{\frac{1}{2}} N^{\varepsilon}\left(\frac{qT}{N}\right)^{\frac{3}{2}+M-\varepsilon} + \frac{T^{\frac{1}{2}}\varphi(q)}{N^{1-\varepsilon}},
\end{aligned}$$

since we had $C = N^D$ for some large but fixed D. Thus, since $qT < N^{1-\varepsilon}$, taking M large enough we have

$$|E(\alpha, \chi)| + |X_\alpha B_{-1}(\alpha, \chi)| \ll \frac{T^{\frac{1}{2}}\varphi(q)}{N^{1-\varepsilon}},$$

where X_α was defined in (4.2.3).

Thus, by (4.2.6) and (4.2.7), to conclude the proof of Theorem 4.1.1 we only need to extract the main term from $V_\alpha\left(\frac{1}{qN}\right)$. This can be done easily by moving the line of integration to $\Re(s) = -1 + \varepsilon$ in (4.2.3). By (4.2.4) we get

$$V_\alpha\left(\frac{1}{qN}\right) = 1 + O\left(\frac{1}{(qNT)^{1-\varepsilon}}\right)$$

and Theorem 4.1.1 follows.

Chapter 5

The first moment of quadratic Dirichlet L-functions

5.1 Introduction

Every real non-principal character (modulo $|d|$) can be written as

$$\chi_d(n) = \left(\frac{d}{n}\right),$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol and d is in the set of quadratic discriminants

$$\mathcal{D} := \{d \in \mathbb{Z} \mid d \neq \square, d \equiv 0, 1 \pmod 4\} \tag{5.1.1}$$

and $\square$ denotes square integers. The character χ_d is primitive if d is a fundamental discriminant, i.e. if d is in the set $\mathcal{F}_0 := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, where

$$\begin{aligned}
\mathcal{F}_1 &= \{d \in \mathbb{Z} \mid d \text{ square-free, } d \equiv 1 \pmod 4\},\\
\mathcal{F}_2 &= \{d \in 4\mathbb{Z} \mid \frac{d}{4} \text{ square-free, } \frac{d}{4} \equiv 2 \pmod 4\},\\
\mathcal{F}_3 &= \{d \in 4\mathbb{Z} \mid \frac{d}{4} \text{ square-free, } \frac{d}{4} \equiv 3 \pmod 4\}.
\end{aligned}$$

If $d \neq 1$ is a fundamental discriminant, the Dirichlet L-function associated to χ_d,

$$L(s, \chi_d) := \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} \qquad \Re(s) > 0,$$

can be analytically continued to an entire function and satisfies the functional equation

$$\Lambda(s,\chi_d) := \left(\frac{|d|}{\pi}\right)^{\frac{s+\kappa}{2}} \Gamma\left(\frac{s+\kappa}{2}\right) L(s,\chi_d) = \Lambda(1-s,\chi_d), \tag{5.1.2}$$

where $\kappa := \frac{1-\chi_d(-1)}{2}$ is 0 if d is positive and 1 if d is negative.

Properties of L-functions associated to real characters have important applications in many directions in number theory. For example, estimates for $L(1,\chi_d)$ can be transferred, via the class number formula, into estimates for the class number of the quadratic field of discriminant d.

An important open problem on quadratic Dirichlet L-functions is to determine the asymptotics for the k-th moment,

$$M^{\pm}_{k,j}(X) := \sum_{\substack{|d|\le X,\\ d\in\mathcal{F}^{\pm}_j}} L\left(\frac{1}{2},\chi_d\right)^k,$$

for $k\in\mathbb{N}$, $j\in\{0,1,2,3\}$, and where $\mathcal{F}^{+}_j$ (respectively $\mathcal{F}^{-}_j$) denotes the positive (resp. negative) elements in $\mathcal{F}_j$.

The families $\mathcal{F}^{\pm}_j$ are symplectic and therefore we expect that

$$M^{\pm}_{k,j}(X) = XP^{\pm}_{k,j}(\log X) + o(X), \tag{5.1.3}$$

where $P^{\pm}_{k,j}$ is a polynomial of degree $\frac{k(k+1)}{2}$ (see Conrey and Farmer [CF], Keating and Snaith [KeS00b], Diaconu et al. [DGH] and Conrey et al. [CFKRS]).

In the case $k=1$ this has been proved by Jutila [Jut81] and Vinogradov and Takhtadzhyan [VT]. They showed that

$$M^{\pm}_{1,0}(X) = \frac{R(1)}{4\zeta(2)}X\left(\log\frac{X}{\pi} + \frac{\Gamma'}{\Gamma}\left(\frac{2\pm 1}{4}\right) + 4\gamma - 1 + 4\frac{R'}{R}(1)\right) + O\left(X^{\frac{3}{4}+\varepsilon}\right), \tag{5.1.4}$$

where γ is Euler's constant and, for $\Re(s)>0$,

$$R(s) := \prod_p\left(1-\frac{1}{(p+1)p^s}\right).$$

The error term in (5.1.4) has been improved to $O\Big(X^{\frac{19}{32}+\varepsilon}\Big)$ by Goldfeld and Hoffstein [GH] and, when a smoothing factor is added, to $O\Big(X^{\frac{1}{2}+\varepsilon}\Big)$ by Young [You09], for $\mathcal{F}_3^{\pm}$ (in this paper it is also claimed that an error term of the same size is essentially implicit in [GH]). Goldfeld and Hoffstein conjecture that the correct order of the error term should be $x^{\frac{1}{4}+\varepsilon}$, but observe that an improvement on the bound $O\Big(X^{\frac{1}{2}}\Big)$ would probably require a major improvement in the zero-free region for the Riemann zeta-function, because of the square-free condition in $\mathcal{F}$.

The asymptotics for $M_{k,0}^{\pm}$ in the case $k = 2$ was proved by Jutila [Jut81], for the leading term, and Soundararajan [Sou00], for the remaining main terms (together with a power saving in the error term). The asymptotics for $k = 3$ has also been proved by Soundararajan (for $\mathcal{F}_3^{+}$), whereas no asymptotics are known for larger values of k.

For the identification of the full main term in (5.1.3), it is particularly useful to consider shifted moments

$$M_{k,j}^{\pm}(X, \alpha_1, \dots, \alpha_k) := \sum_{\substack{|d| \le X, \\ d \in \mathcal{F}_j^{\pm}}} L\bigg(\frac{1}{2} + \alpha_1, \chi_d\bigg) \cdots L\bigg(\frac{1}{2} + \alpha_k, \chi_d\bigg).$$

Young [You09, You1] proved the asymptotics for these averages (for $\mathcal{F}_3^{+}$ and with a smoothing factor) in the case $k = 1, 3$ for shift parameters in the range $\Re(\alpha) \ll \frac{1}{\log X}$, $\Im(\alpha) \ll X^{\varepsilon}$. (In [GH], Goldfeld and Hoffstein allow for a bounded shift with positive real part).

For some applications it is useful to allow the shifts to be large in the imaginary direction. For example, when computing the one-level density for zeros of $L\big(\frac{1}{2}, \chi_d\big)$, Conrey and Snaith [CS] are led to integrate the ratios conjecture over the shifts on an unbounded interval.

Some results in this direction have been obtained by Jutila [Jut75] and Heath-Brown [H-B95]. They showed respectively that

$$\int_{-T}^{T} M_{2,0}(X, it, it) \ll XT \log^{16}(X(T+2)),$$

and

$$M_{4,0}(X, it, it, it, it) \ll (X(|t|+1))^{1+\varepsilon}.$$

Note also that by the Cauchy Schwartz inequality, the latter result implies that

$$M_{1,0}(X, it) \ll (|t|+1)^{\frac{1}{4}+\varepsilon} X^{1+\varepsilon}.$$

In this chapter we consider the first moment $M^{+}_{1,3}(\alpha, X)$, showing that the asymptotic formula is valid in the range $\Im(\alpha) \ll X^{\frac{3}{5}-\varepsilon}$, $\Re(\alpha) \ll \frac{1}{\log X}$.

Theorem 5.1.1. *Let* $\zeta_2(s) := (1-2^{-s})\,\zeta(s)$ *and*

$$Q(s) := \frac{2}{3}\prod_{p\neq 2}\left(1-\frac{1}{p^{s+1}-p^{s}+1}\right)^{-1}.$$

Then for $X \geq 3$ *and* $\Re(\alpha) \ll \frac{1}{\log X}$ *we have*

$$\begin{aligned} M^{+}_{1,3}(X,\alpha) = X\frac{\zeta(1+2\alpha)}{8\zeta(2)}\,Q(1+2\alpha)+ & \\ +\frac{\Gamma\left(\frac{\frac{1}{2}-\alpha}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)}\frac{\pi^{\alpha}X^{1-\alpha}}{1-\alpha}\frac{\zeta_2(1-2\alpha)}{8\zeta(2)}\,Q(1-2\alpha)+E(X,\alpha), \end{aligned} \tag{5.1.5}$$

where

$$E(X,\alpha) \ll \begin{cases} (|\Im(\alpha)|+1)^{\frac{3}{4}}X^{\frac{3}{4}}\log^{\frac{5}{2}} X & \textit{if } |\Im(\alpha)| \leq X^{\frac{1}{9}}, \\ |\Im(\alpha)|^{\frac{3}{10}}X^{\frac{4}{5}+\varepsilon} & \textit{if } T^{\frac{1}{9}} \leq |\Im(\alpha)| \leq X^{\frac{1}{4}}, \\ |\Im(\alpha)|^{\frac{5}{14}}X^{\frac{11}{14}+\varepsilon} & \textit{if } |\Im(\alpha)| \geq X^{\frac{1}{4}}. \end{cases}$$

If $\alpha = 0$ *the left hand side is to be interpreted as the limit as* $\alpha \to 0$.

An adaptation of the work of Jutila [Jut81] to the shifted case gives the bound $(|\Im(\alpha)|+1)^{\frac{3}{4}}X^{\frac{3}{4}+\varepsilon}$ for the error term (which is smaller than the main term if $\Im(\alpha) \ll X^{\frac{1}{3}-\varepsilon}$). When $|\Im(\alpha)| \geq X^{\frac{1}{9}}$ this bound can be improved by using a result of Heath-Brown [H-B78] on character sums of the form $\sum_{n\approx M}\chi(n)n^{-it}$.

5.2 Preliminaries

In what follows we will assume

$$T := |\Im(\alpha)| > 2, \qquad \Re(\alpha) \ll \frac{1}{\log X}. \tag{5.2.1}$$

The condition $T > 2$ can be then removed by the maximum modulus principle, since, by (5.1.5), $E(\alpha, X)$ is holomorphic near 0. We will also assume $T \ll X^{100}$ as the result is trivial for larger T.

We need to compute the asymptotics of

$$M_{1,3}(8X, \alpha) := \sum_{\substack{(2,d)=1,\\ 0<d\leq X}}^{*} L\left(\frac{1}{2} + \alpha, \chi_{8d}\right),$$

where here and later $\sum^{*}$ indicates a sum over square-free numbers. A good strategy is to start by expressing $L\left(\frac{1}{2}, \chi_{8d}, \alpha\right)$ in terms of "short sums". To do this we use the approximate functional equation, which allows us to express $L\left(\frac{1}{2}, \chi_{8d}, \alpha\right)$ as a sum of two series whose contributions come essentially from the first $O\left(Y d^{\frac{1}{2}+\varepsilon}\right)$ and $O\left(Y^{-1} d^{\frac{1}{2}+\varepsilon}\right)$ terms, where Y is a parameter that we will choose later.

Lemma 5.2.1. *Let $G(s)$ be an even function which grows at most as a polynomial on vertical strips and assume $G(0) = 1$. Then, for any $Y > 0$ we have*

$$L\left(\frac{1}{2} + \alpha, \chi_d\right) = \sum_n \frac{\chi_{8d}(n)}{n^{\frac{1}{2}+\alpha}} V_\alpha\left(Y \frac{n}{d^{\frac{1}{2}}}\right) + X_\alpha \sum_n \frac{\chi_{8d}(n)}{n^{\frac{1}{2}-\alpha}} d^{-\alpha} V_{-\alpha}\left(\frac{1}{Y} \frac{n}{d^{\frac{1}{2}}}\right), \tag{5.2.2}$$

where

$$V_\alpha(x) := \frac{1}{2\pi i} \int_{(1)} G(s) \frac{\Gamma\left(\frac{\frac{1}{2}+\alpha+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)} \left(\frac{\pi x^2}{8}\right)^{-\frac{s}{2}} \frac{\mathrm{d}s}{s}$$

and

$$X_\alpha := \left(\frac{8}{\pi}\right)^{-\alpha} \frac{\Gamma\left(\frac{\frac{1}{2}-\alpha}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)}.$$

Proof. Using the integral representation of V_α we get

$$\begin{aligned}\sum_n \frac{\chi_d(n)}{n^{\frac{1}{2}+\alpha}} V_\alpha\left(Y\frac{n}{d^{\frac{1}{2}}}\right) = \frac{1}{2\pi i}\int_{(1)} G(s)\frac{\Gamma\left(\frac{\frac{1}{2}+\alpha+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)} L\left(\frac{1}{2}+\alpha+s,\chi_d\right)\times \\ \times\left(\frac{\pi Y}{8d}\right)^{-\frac{s}{2}}\frac{\mathrm{d}s}{s}.\end{aligned}$$

Moving the line of integration to $\Re(s)=-1$, we encounter a simple pole at 0 of residue $L\left(\frac{1}{2}+\alpha,\chi_d\right)$, thus, after the change of variable $s\to -s$, we have

$$\begin{aligned}\sum_n \frac{\chi_d(n)}{n^{\frac{1}{2}+\alpha}} V_\alpha\left(Y\frac{n}{d^{\frac{1}{2}}}\right) = \frac{1}{2\pi i}\int_{(1)} G(s)\frac{\Gamma\left(\frac{\frac{1}{2}+\alpha-s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)} L\left(\frac{1}{2}+\alpha-s,\chi_d\right)\times \\ \times\left(\frac{\pi Y}{8d}\right)^{\frac{s}{2}}\frac{\mathrm{d}s}{s}+L\left(\frac{1}{2}+\alpha,\chi_d\right)\end{aligned}$$

and the lemma follows after applying the functional equation (5.1.2). ☐

Remark 5.2.2. *It turns out it is useful for our computation that we take a function $G(s)$ that decays fast on vertical strips and with zeros at $s=\pm\alpha$ and $s=\pm\frac{1}{2}\pm\alpha$. To make this explicit we take*

$$G(s) := \frac{(s^2-\alpha^2)\left(s^2-\left(\frac{1}{2}+\alpha\right)^2\right)\left(s^2-\left(\frac{1}{2}-\alpha\right)^2\right)}{\alpha^2(\frac{1}{2}+\alpha)^2(\frac{1}{2}-\alpha)^2}\,e^{s^2}.$$

With this choice of $G(s)$, if $\sigma := \Re(s)$ is bounded we have

$$\begin{aligned}G(s)\frac{\Gamma\left(\frac{\frac{1}{2}+\alpha+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)} &\ll T^{\frac{\sigma}{2}}e^{-\frac{\pi}{4}(|s+\alpha|-|\alpha|)}|G(s)| \\ &\ll T^{\frac{\sigma}{2}}e^{-\frac{|s|^2}{2}},\end{aligned} \tag{5.2.3}$$

for α satisfying (5.2.1) *and $s+\alpha+\frac{1}{2}-2m\gg 1$ for all non-negative integer m. Thus,*

$$V_\alpha(x) \ll T^{\frac{B}{2}}x^{-B}, \tag{5.2.4}$$

for all fixed $B>0$.

Remark 5.2.3. *Throughout the rest of the chapter we will assume* $1 \leq Y \ll X^{100}$.

Using the approximate functional equation (5.2.2), we can write $M_{1,3}(8X,\alpha)$ as

$$M_{1,3}(8X,\alpha) = S_0(X,\alpha) + X_\alpha S_1(X,-\alpha), \tag{5.2.5}$$

where

$$S_\epsilon(X,\alpha) := \sum_{n\geq 1} \frac{1}{n^{\frac{1}{2}+\alpha}} \sideset{}{^*}\sum_{\substack{(2,d)=1,\\ d\leq X}} \chi_{8d}(n) d^{\epsilon\alpha} V_\alpha\Big(Y^{2\epsilon-1}\frac{n}{d^{\frac{1}{2}}}\Big),$$

with $\epsilon = 0, 1$.

It is reasonable to expect that the sum of those n which are square contributes a main term,

$$M_{\square,\epsilon}(X,\alpha) := \sum_{\substack{m\geq 1,\\ (m,2)=1}} \frac{1}{m^{1+2\alpha}} \sideset{}{^*}\sum_{\substack{d\leq X,\\ (d,2m)=1}} d^{\epsilon\alpha} V_\alpha\Big(Y^{2\epsilon-1}\frac{m^2}{d^{\frac{1}{2}}}\Big),$$

whereas the sum of the non-square n,

$$\begin{aligned} E_\epsilon(X,\alpha) &:= S_\epsilon(X,\alpha) - M_{\square,\epsilon}(X,\alpha) \\ &= \sum_{\substack{n\geq 1,\\ n\neq\square}} \frac{1}{n^{\frac{1}{2}+\alpha}} \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq X}} \chi_{8d}(n) d^{\epsilon\alpha} V_\alpha\Big(Y^{2\epsilon-1}\frac{n}{d^{\frac{1}{2}}}\Big), \end{aligned} \tag{5.2.6}$$

gives an error term. (Notice that this intuition is not true for the second and third moment as results from Soundararajan's computations [Sou00]).

5.3 Computing the main terms

Using the integral representation of V_α we can write $M_{\square,\epsilon}(X,\alpha)$ as

$$\begin{aligned} M_{\square,\epsilon}(X,\alpha) = \frac{1}{2\pi i}\int_{(1)} G(s) \frac{\Gamma\Big(\frac{\frac{1}{2}+\alpha+s}{2}\Big)}{\Gamma\Big(\frac{\frac{1}{2}+\alpha}{2}\Big)} \Big(\frac{\pi}{8}\Big)^{-\frac{s}{2}} Y^{(1-2\epsilon)s} \times \\ \times F\Big(X, \epsilon\alpha + \frac{s}{2}, 1+2\alpha+2s\Big)\frac{\mathrm{d}s}{s}, \end{aligned} \tag{5.3.1}$$

where, for $\Re(z) > 1$,

$$F(X,s,z) := \sum_{\substack{m\geq 1,\\ (m,2)=1}} \sideset{}{^*}\sum_{\substack{d\leq X,\\ (d,2m)=1}} \frac{d^s}{m^z}.$$

We start by computing the inner sum in $F(X,s,z)$.

Lemma 5.3.1. *Let $s = \sigma + it$, with $\sigma > 0$, and let $\tau = |t| + 2$. Then*

$$\sideset{}{^*}\sum_{\substack{(d,2m)=1,\\ d\leq X}} d^s = \frac{a_{2m}}{\zeta(2)}\frac{X^{1+s}}{1+s} + O\left(d(2m)\frac{X^{\frac{1}{2}+\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma}\right),$$

where $d(m)$ is the divisor function and

$$a_m := \prod_{p|m}\left(1+\frac{1}{p}\right)^{-1}.$$

Proof. By Möbius inversion we have $\mu(d)^2 = \sum_{\ell^2|d}\mu(\ell)$ and thus

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{(d,2m)=1,\\ d\leq X}} d^s &= \sum_{\substack{(d,2m)=1,\\ d\leq X}} \mu(d)^2 d^s = \sum_{\substack{(d,2m)=1,\\ d\leq X}}\sum_{\ell^2|d}\mu(\ell)d^s \\ &= \sum_{\substack{(\ell,2m)=1,\\ \ell\leq\sqrt{X}}}\mu(\ell)\ell^{2s}\sum_{\substack{(a,2m)=1,\\ a\leq\frac{X}{\ell^2}}} a^s.\end{aligned} \tag{5.3.2}$$

Moreover,

$$\sum_{\substack{(a,n)=1,\\ a\leq Y}} a^s = \sum_{\substack{a\leq Y,\\ (a,n)=1}}\sum_{c|(a,n)}\mu(c)a^s = \sum_{c|n}\mu(c)c^s\sum_{a\leq\frac{Y}{c}} a^s.$$

Now, for $t > 0$ we apply Lemma 4.7 in Titchmarsh [Tit] to the function $f(x) = \frac{t}{2\pi}\log x$ and we get that

$$\sum_{n\leq X} n^{it} = \sum_{\frac{|t|}{2\pi X}-\frac{1}{2\pi}\leq\nu\leq\frac{|t|}{2\pi}+\frac{1}{2\pi}}\int_1^X y^{it}e^{2\pi i\nu y}\,\mathrm{d}y + O(\log\tau). \tag{5.3.3}$$

Now, if $\nu > 0$ we have

$$\int_1^X y^{it}e^{-2\pi i\nu y}\,\mathrm{d}y = \frac{1}{\nu^{1-it}}\int_\nu^{\nu X} y^{it}e^{-2\pi iy}\,\mathrm{d}y \ll \frac{\sqrt{\tau}}{\nu}, \tag{5.3.4}$$

by Lemma 4.8 in Titchmarsh (and where we have evaluated trivially the interval of length $2\sqrt{\tau}$ centered at $t/2\pi$). The term $\nu = 0$ is equal to

$$\int_1^X y^{it}\,\mathrm{d}y = \frac{X^{1+it}}{1+it} + O(1)$$

and is $O(1)$ if $X \geq t$. Thus, from (5.3.3) and (5.3.4) we have

$$\sum_{n\leq X} n^{it} = \frac{X^{1+it}}{1+it} + O\big(1+\sqrt{\tau}\log\tau\big) \tag{5.3.5}$$

and, by taking the conjugate, (5.3.5) clearly holds also for $t \leq 0$. Thus, by partial summation, for $\sigma > 0$ one has

$$\sum_{n\leq X} n^{s} = \frac{X^{1+s}}{1+s} + O\bigg(\frac{X^{\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma}\bigg).$$

It follows that

$$\begin{aligned}\sum_{\substack{(a,n)=1,\\ a\leq Y}} a^{s} &= \sum_{c|n}\frac{\mu(c)}{c}\frac{Y^{1+s}}{1+s} + O\bigg(d(n)\frac{Y^{\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma}\bigg)\\ &= \frac{\varphi(n)}{n}\frac{Y^{1+s}}{1+s} + O\bigg(d(n)\frac{Y^{\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma}\bigg),\end{aligned}$$

where $\varphi(n)$ is Euler's totient function. Inserting this formula in (5.3.2), we get

$$\sideset{}{^*}\sum_{\substack{(d,2m)=1,\\ d\leq X}} d^{s} = \frac{\varphi(2m)}{2m}\frac{X^{1+s}}{1+s}\sum_{\substack{(\ell,2m)=1,\\ \ell\leq\sqrt{X}}}\frac{\mu(\ell)}{\ell^2} + O\bigg(d(2m)\frac{X^{\frac{1}{2}+\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma}\bigg)$$

and the lemma follows. $\square$

Using the previous lemma, we can give the following approximation for $F(X,s,z)$.

Lemma 5.3.2. *Let $s = \sigma + it$, with $\sigma > 0$ and $\tau = |t| + 2$. Let $y = \Re(z) > 1$. We have*

$$F(X,s,z) = \frac{1}{\zeta(2)}\frac{X^{1+s}}{1+s}\zeta_2(z)\,Q(z) + O\Big(\frac{y^2X^{\frac{1}{2}+\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma(y-1)^2}\Big),$$

where

$$Q(z) := \frac{2}{3}\prod_{p\neq 2}\bigg(1-\frac{1}{p^{z+1}-p^{z}+1}\bigg)^{-1}$$

is analytic and bounded in $\Re(z) > \delta$ for all fixed $\delta > 0$.

Proof. By Lemma 5.3.1 we have

$$\begin{aligned}
F(X,s,z) &= \frac{1}{\zeta(2)}\frac{X^{1+s}}{1+s}\sum_{\substack{m\geq 1,\\(m,2)=1}}\frac{a_{2m}}{m^z} + O\Big(\frac{X^{\frac{1}{2}+\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma}\sum_{\substack{m\geq 1,\\(m,2)=1}}\frac{d(2m)}{m^y}\Big)\\
&= \frac{1}{\zeta(2)}\frac{X^{1+s}}{1+s}\sum_{\substack{m\geq 1,\\(m,2)=1}}\frac{a_{2m}}{m^z} + O\Big(\frac{y^2X^{\frac{1}{2}+\sigma}\tau^{\frac{1}{2}}\log\tau}{\sigma(y-1)^2}\Big).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\sum_{(n,2)=1}\frac{a_{2n}}{n^s} &= \frac{2}{3}\prod_{p\neq 2}\Big(1+a_p\frac{1}{p^s-1}\Big) = \frac{2}{3}\prod_{p\neq 2}\Big(1-\frac{1}{p^s-p^{s-1}+\frac{1}{p}}\Big)^{-1}\\
&= \zeta_2(s)Q(s)
\end{aligned}$$

and the result follows. □

Moving the line of integration in (5.3.1) to $\Re(s)=\varepsilon=\frac{1}{\log X}$ and applying Lemma 5.3.2, we get

$$M_{\square,\epsilon}(X,\alpha)=\mathcal{M}_{\square,\epsilon}(X,\alpha)+\mathcal{E}_{\square,\epsilon}(X,\alpha), \tag{5.3.6}$$

where

$$\begin{aligned}
\mathcal{M}_{\square,\epsilon}(X,\alpha) := \frac{1}{2\pi i}\int_{(\varepsilon)} G(s)\frac{\Gamma\Big(\frac{\frac{1}{2}+\alpha+s}{2}\Big)}{\Gamma\Big(\frac{\frac{1}{2}+\alpha}{2}\Big)}\Big(\frac{\pi}{8}\Big)^{-\frac{s}{2}}\frac{Y^{(1-2\epsilon)s}}{\zeta(2)}\frac{X^{1+\epsilon\alpha+\frac{s}{2}}}{1+\epsilon\alpha+\frac{s}{2}}\times\\
\times\,\zeta_2(1+2\alpha+2s)\,Q(1+2\alpha+2s)\frac{\mathrm{d}s}{s}
\end{aligned} \tag{5.3.7}$$

and, by (5.2.3),

$$\mathcal{E}_{\square,\epsilon}(X,\alpha)\ll (XT)^{\frac{1}{2}}\log^4 X\log\log X. \tag{5.3.8}$$

Now we move the line of integration in (5.3.7) to $\Re(s)=-\frac{1}{2}$. We encounter a pole at $s=0$ (notice that the poles in $s=-\frac{1}{2}-\alpha$ and $s=-\alpha$ are canceled by the zeros of $G(s)$) and thus, by (5.2.3), we have

$$\mathcal{M}_{\square,\epsilon}(X,\alpha)=\frac{X^{1+\epsilon\alpha}}{1+\epsilon\alpha}\frac{\zeta_2(1+2\alpha)}{\zeta(2)}\,Q(1+2\alpha)+O\Big(Y^{-\frac{1}{2}+\epsilon}T^{\frac{1}{4}-\epsilon}X^{\frac{3}{4}}\Big). \tag{5.3.9}$$

Therefore, by (5.3.6)-(5.3.9), we have that the contribution of $M_{\Box,0}$ and $M_{\Box,1}$ to $M_{1,3}(8X,\alpha)$ is (cf. (5.2.5))

$$\begin{aligned}M_{\Box,0}(X,\alpha)+&X_\alpha M_{\Box,1}(X,-\alpha) = \\ &= X\frac{\zeta_2(1+2\alpha)}{\zeta(2)}\,Q(1+2\alpha)+X_\alpha\frac{X^{1-\alpha}}{1-\alpha}\frac{\zeta_2(1-2\alpha)}{\zeta(2)}\,Q(1-2\alpha)+ \\ &\quad+O\biggl((XT^{\epsilon})^{\frac{1}{2}}\log^4 X\log\log X+\max_{\epsilon=0,1}\Bigl(Y^{-\frac{1}{2}+\epsilon}T^{\frac{1}{4}-\epsilon}X^{\frac{3}{4}}\Bigr)\biggr).\end{aligned} \tag{5.3.10}$$

5.4 Bounding the error terms

To bound the error terms we need an estimate for the character sum

$$S(X,s,z):=\sum_{n\neq\Box}\sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\le X}}\frac{\chi_{8d}(n)d^s}{n^z},$$

for $\Re(s)>0$ and $\Re(z)>1$. If $\Im(s)$ is small enough compared to X, we can deduce a useful bound for $S(X,s,z)$ by proceeding in the same way as in [Jut81], starting from the following result which is Theorem 2 of Armon [Arm].

Lemma 5.4.1. *Let $X,Y\geq 2$ and let $\mathcal{D}$ be as defined in* (5.1.1)*, then*

$$\sum_{\substack{|d|\le X,\\ d\in\mathcal{D}}}\left|\sum_{n\le Y}\left(\frac{d}{n}\right)\right|^2\ll XY\log X. \tag{5.4.1}$$

Corollary 5.4.2. *Let $X,N\geq 2$, we have*

$$\sum_{\substack{n\le N,\\ (n,2)=1,\\ n\neq\Box}}\left|\sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\le X}}\chi_{8d}(n)\right|^2\ll XN\log X\log^2 N.$$

Proof. Firstly we remove the square-free condition from the sum over d. By the quadratic reciprocity law for the Kronecker symbol and Möbius inversion formula, for an odd n we have

$$\begin{aligned}
\Big|\sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\le X}} \chi_{8d}(n)\Big| &= \Big|\sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\le X}} \Big(\frac{n}{d}\Big)\Big| = \Big|\sum_{\substack{(m,2)=1,\\ m\le X}} \Big(\frac{n}{m}\Big)\sum_{c^2|m}\mu(c)\Big| \\
&= \Big|\sum_{\substack{(c,2n)=1,\\ c\le\sqrt{X}}} \mu(c) \sum_{\substack{(\ell,2)=1,\\ \ell\le \frac{X}{c^2}}} \Big(\frac{n}{\ell}\Big)\Big| \\
&\ll \sum_{c\le\sqrt{X}}\Big|\sum_{\ell\le\frac{X}{c^2}}\Big(\frac{n}{\ell}\Big)\Big| + \sum_{c\le\sqrt{X}}\Big|\sum_{\ell\le\frac{X}{2c^2}}\Big(\frac{n}{\ell}\Big)\Big|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{\substack{n\neq\Box,\\ (2,n)=1,\\ n\le N}}\Big|\sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\le X}}\chi_{8d}(n)\Big|^2 \ll \sum_{\substack{|n|\le N,\\ n\in\mathcal{D}}}\Bigg(\Big(\sum_{c\le\sqrt{X}}\Big|\sum_{\ell\le\frac{X}{c^2}}\Big(\frac{n}{\ell}\Big)\Big|\Big)^2 \\
+\Big(\sum_{c\le\sqrt{X}}\Big|\sum_{\ell\le\frac{X}{2c^2}}\Big(\frac{n}{\ell}\Big)\Big|\Big)^2\Bigg).
\end{aligned}$$

Now, applying the Cauchy-Schwartz inequality and (5.4.1), we have that

$$\begin{aligned}
\sum_{\substack{|n|\le N,\\ n\in\mathcal{D}}}\Big(\sum_{c\le\sqrt{X}}\Big|\sum_{\ell\le\frac{X}{c^2}}\Big(\frac{n}{\ell}\Big)\Big|\Big)^2 &= \sum_{c_1\le\sqrt{X}}\sum_{c_2\le\sqrt{X}}\sum_{\substack{|n|\le N,\\ n\in\mathcal{D}}}\Big(\sum_{\ell\le\frac{X}{c_1^2}}\Big(\frac{n}{\ell}\Big)\Big)\Big(\sum_{\ell\le\frac{X}{c_2^2}}\Big(\frac{n}{\ell}\Big)\Big) \\
&\ll \Bigg(\sum_{c\le\sqrt{X}}\Big(\sum_{\substack{|n|\le N,\\ n\in\mathcal{D}}}\Big|\sum_{\ell\le\frac{X}{c^2}}\Big(\frac{n}{\ell}\Big)\Big|^2\Big)^{\frac12}\Bigg)^2 \\
&\ll XN\log X\log^2 N
\end{aligned}$$

and the Corollary follows. □

We can now deduce the following bound for $S(X,s,z)$.

Lemma 5.4.3. *Let $s=\sigma+it$, $z=x+iy$, with $\sigma>0$ and $x>1$. Then, for $X\ge 2$, we have*

$$S(X,s,z)\ll\Big(1+\frac{|s|}{\sigma}\Big)\Big(\frac{x}{x-1}\Big)^2 X^{\sigma+\frac12}\sqrt{\log X}. \qquad (5.4.2)$$

Proof. By partial summation we have

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq X}} \chi_{8d}(n)d^s &= X^s \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq X}} \chi_{8d}(n) - s\int_1^X \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq r}} \chi_{8d}(n) r^{s-1}\,\mathrm{d}r\\ &\ll \left(1+\frac{|s|}{\sigma}\right) X^{\sigma} \max_{r\leq X}\left| \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq r}} \chi_{8d}(n)\right|.\end{aligned}$$

Thus, by Cauchy-Schwartz inequality and Lemma 5.4.2, we have

$$\begin{aligned}\sum_{\substack{n\neq\square,\\ n\leq N}} \left| \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq X}} \chi_{8d}(n)d^s\right| &\ll \sqrt{N}\left(\sum_{\substack{n\neq\square,\\ n\leq N}} \left| \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq X}} \chi_{8d}(n)d^s\right|^2\right)^{\frac{1}{2}}\\ &\ll \left(1+\frac{|s|}{\sigma}\right) X^{\sigma+\frac{1}{2}} N\log N\sqrt{\log X}.\end{aligned}$$

Therefore,

$$\begin{aligned}S(X,s,z) &\ll \sum_{N=0}^{\infty} \frac{1}{2^{Nx}} \sum_{\substack{n\neq\square,\\ 2^N\leq n<2^{N+1}}} \left| \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq X}} \frac{\chi_{8d}(n)d^s}{n^{1+z}}\right|\\ &\ll \left(1+\frac{|s|}{\sigma}\right)\left(\frac{x}{x-1}\right)^2 X^{\sigma+\frac{1}{2}}\sqrt{\log X}.\end{aligned}$$

□

If $\Im(s)$ is bigger than $X^{\frac{1}{3}}$ the bound (5.4.2) is not good enough for our purposes. We will deduce a new bound for

$$S(X,A,s,z) := \sum_{\substack{n\neq\square,\\ n\leq A}} \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ d\leq X}} \frac{\chi_{8d}(n)d^s}{n^z}$$

from the following lemma, which is due to Heath-Brown [H-B78].

Lemma 5.4.4. *Let χ be a non-principal character modulo q and let*

$$M(N,\chi) := \max_{M\leq N}\left| \sum_{N\leq n\leq N+M} \frac{\chi(n)}{n^{it}}\right|.$$

Then, writing $\tau=|t|+2$, one has

$$\begin{aligned}M(N) \ll N^{\frac{1}{2}}\left(d(q)\,\sigma_{-\frac{1}{4}}(q)\right)^{\frac{8}{3}}\left(q_0^{\frac{1}{2}}+\left(\frac{q}{q_0}\right)^{\frac{1}{4}}\log^{\frac{1}{2}}(q\tau)+(q\tau)^{\frac{1}{6}}\right)\times\\ \times\left(1+\frac{N^{\frac{1}{2}}}{(q\tau)^{\frac{1}{4}}}+\frac{N^{\frac{7}{16}}}{(q\tau)^{\frac{7}{32}}}\log^{\frac{1}{2}}(q\tau)\right),\end{aligned} \tag{5.4.3}$$

for any divisor q_0 of q and where, as usual, $\sigma_a(n) := \sum_{d|n} n^a$.

Proof. If χ is primitive, then the Lemma is the inequality in the last display of page 168 in [H-B78]. (At the beginning of Heath-Brown's paper it is assumed that $N \le (q\tau)^{\frac{1}{2}+\varepsilon}$, however this condition is not used to prove the inequality (5.4.3)). The result can be then extended to all non principal characters by observing that if χ induced by a primitive character χ_1 modulo $q_1 > 1$, then writing $r := q/q_1$, one has

$$\sum_{N\le n\le 2N} \frac{\chi(n)}{n^{it}} = \sum_{N\le n\le N+M} \frac{\chi_1(n)}{n^{it}} \sum_{\ell|(n,r)} \mu(\ell) = \sum_{\ell|r} \mu(\ell)\chi_1(\ell) \sum_{N\le na\le N+M} \frac{\chi_1(n)}{n^{it}}$$
$$\le d(r) \Bigg| \sum_{\frac{N}{a}\le n\le 2\frac{N}{a}} \frac{\chi_1(n)}{n^{it}} \Bigg| \le d(r) \max_{a|r} M\left(\frac{N}{a}\right),$$

from which the lemma follows easily. □

Corollary 5.4.5. *Let $\Re(z) = x < \frac{7}{6} - \varepsilon$ and let $s = \sigma + it$, with $\Re(s) > -\frac{1}{2} + \varepsilon$, for some $\varepsilon > 0$. Moreover, let $\tau = |t| + 2$. Then, if $X \ll \tau^C$ for some fixed constant C, then*

$$S(X, M, s, z) \ll \left(M^{\frac{5}{4}-x+\varepsilon} \log^{\frac{1}{2}}(|t|+2) + (|t|+2)^{\frac{1}{6}} M^{\frac{7}{6}-x+\varepsilon}\right) X^{\frac{1}{2}+\sigma}.$$

Proof. Proceeding as in the proof of Corollary 5.4.2, we have that

$$\sum_{\substack{n\ne\square,\\ (2,n)=1,\\ N\le n\le 2N}} \Bigg| \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ X\le d\le 2X}} \chi_{8d}(n) d^s \Bigg| \ll X^\sigma \sum_{\substack{N\le |n|\le 2N,\\ n\in\mathcal{D}}} \max_{j=1,2} \Bigg(\sum_{c\le\sqrt{2X}} \Bigg| \sum_{\frac{X}{jc^2}\le \ell\le \frac{2X}{jc^2}} \left(\frac{n}{\ell}\right) \ell^{it} \Bigg| \Bigg).$$

Now, since $\sigma_{-\frac{1}{4}}(n), d(n) \ll_\varepsilon n^{\frac{3}{32}\varepsilon}$, using Lemma 5.4.4 with $q_0 = 1$ we have that

$$\sum_{c\le\sqrt{2X}} \Bigg| \sum_{\frac{X}{jc^2}\le \ell\le \frac{2X}{jc^2}} \left(\frac{n}{\ell}\right) \ell^{it} \Bigg| \ll n^{\frac{\varepsilon}{2}} X^{\frac{1}{2}} \left(n^{\frac{1}{4}} \log^{\frac{1}{2}}(n\tau) + (n\tau)^{\frac{1}{6}}\right) \times$$
$$\times \left(1 + \frac{\log X}{(n\tau)^{\frac{1}{4}}} + \frac{1}{(n\tau)^{\frac{7}{32}}} \log^{\frac{1}{2}}(n\tau)\right).$$

Thus, for $x < \frac{7}{6} - \varepsilon$ one has

$$S(X, M, s, z) - S(X/2, M, s, z) \ll \sum_{N=0}^{[\log_2 M]+1} \Bigg| \sum_{\substack{n\ne\square,\\ (2,n)=1,\\ 2^N\le n\le 2^{N+1}}} \sideset{}{^*}\sum_{\substack{(d,2)=1,\\ \frac{X}{2}\le d\le X}} \frac{\chi_{8d}(n) d^s}{n^z} \Bigg|$$
$$\ll \left(M^{\frac{5}{4}-x+\varepsilon} \log^{\frac{1}{2}} \tau + \tau^{\frac{1}{6}} M^{\frac{7}{6}-x+\varepsilon}\right) X^{\frac{1}{2}+\sigma}$$

and the Corollary follows. □

We can now use these results to bound the error terms. Firstly we notice that using the integral representation of V_α we can write $E_\epsilon(X,\alpha)$ from (5.2.6) as

$$\begin{aligned} E_\epsilon(X,\alpha) = \frac{1}{2\pi i}\int_{(1)} G(s)\frac{\Gamma\left(\frac{\frac{1}{2}+\alpha+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)}\left(\frac{\pi}{8}\right)^{-\frac{s}{2}} Y^{(1-2\epsilon)s}\times \\ \times S\left(X, \epsilon\alpha+\frac{s}{2}, \frac{1}{2}+\alpha+s\right)\frac{\mathrm{d}s}{s}. \end{aligned} \tag{5.4.4}$$

Applying Lemma 5.4.3, we get the following bound.

Lemma 5.4.6. *For $\epsilon\in\{0,1\}$, we have*

$$E_\epsilon(X,\alpha) \ll T^{\frac{1}{4}+\epsilon}Y^{\frac{1}{2}-\epsilon}X^{\frac{3}{4}}\log^{\frac{5}{2}} X. \tag{5.4.5}$$

Proof. We move the line of integration in (5.4.4) to $\Re(s)=\frac{1}{2}+\delta$ and apply Lemma 5.4.3 and (5.2.3), getting

$$\begin{aligned} E_\epsilon(X,\alpha) &= \frac{1}{2\pi i}\int_{\left(\frac{1}{2}+\delta\right)} G(s)\frac{\Gamma\left(\frac{\frac{1}{2}+\alpha+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)}\left(\frac{\pi}{8}\right)^{-\frac{s}{2}} Y^{(1-2\epsilon)s}\times \\ &\qquad \times S\left(X, \epsilon\alpha+\frac{s}{2}, \frac{1}{2}+\alpha+s\right)\frac{\mathrm{d}s}{s} \\ &\ll \frac{1}{\delta^2}T^{\frac{1}{4}+\delta}|1+\epsilon\alpha|Y^{(1-2\epsilon)\left(\frac{1}{2}+\delta\right)}X^{\frac{3}{4}+\delta}\sqrt{\log X} \end{aligned}$$

and the lemma follows by taking $\delta=\frac{1}{\log X}$. □

We now derive another bound for $E_1(X,\alpha)$ which is smaller than (5.4.5) if T is large.

Lemma 5.4.7. *We have*

$$E_1(X,\alpha) \ll T^{\frac{3}{8}}Y^{-\frac{3}{4}}X^{\frac{7}{8}+\varepsilon} + T^{\frac{1}{2}}Y^{-\frac{2}{3}}X^{\frac{5}{6}+\varepsilon}.$$

Proof. Firstly, we observe that by (5.2.4) we can truncate the sum over n at height $A := Y^{-1}(|1+\alpha|X)^{\frac{1}{2}}X^{\varepsilon}$, for any $\varepsilon > 0$. In fact, by (5.2.4) we have that for all $B > 0$ the contribution for the terms $n \geq A$ is bounded by

$$XA^{B}X^{-\varepsilon B}\sum_{\substack{n>A,\\ n\neq\square}}\frac{1}{n^{\frac{1}{2}+\Re(\alpha)+B}} \ll A^{\frac{1}{2}}X^{1-\varepsilon B} = O(1),$$

if B is large enough. Thus, by the integral representation of V_α, it follows that

$$E_1(X,\alpha) = \frac{1}{2\pi i}\int_{(1)} G(s)\frac{\Gamma\Big(\frac{\frac{1}{2}+\alpha+s}{2}\Big)}{\Gamma\Big(\frac{\frac{1}{2}+\alpha}{2}\Big)}\Big(\frac{\pi}{8}\Big)^{-\frac{s}{2}}Y^{-s}\times$$
$$\times S\Big(X, A, \alpha+\frac{s}{2}, \frac{1}{2}+\alpha+s\Big)\frac{\mathrm{d}s}{s} + O(1).$$

Moving the line of integration to $\Re(s) = \frac{1}{2}$, we get, by Lemma 5.4.5 and (5.2.3),

$$E_1(X,\alpha) \ll X^{\frac{1}{2}+\varepsilon}\Big(A^{\frac{3}{4}}\log^{\frac{1}{2}}T + A^{\frac{2}{3}}T^{\frac{1}{6}}\Big),$$

which proves the lemma. ☐

5.5 Proof of the Theorem

Firstly we observe that Lemma 5.4.6, together with (5.3.10), gives Theorem 5.1.1 with an error

$$E(\alpha,X) \ll \max_{\epsilon=0,1}\Big(Y^{-\frac{1}{2}+\epsilon}T^{\frac{1}{4}-\epsilon}X^{\frac{3}{4}} + T^{\frac{1}{4}+\epsilon}Y^{\frac{1}{2}-\epsilon}X^{\frac{3}{4}}\log^{\frac{5}{2}}X + T^{\frac{\epsilon}{2}}X^{\frac{1}{2}+\varepsilon}\Big).$$

The choice $Y = T$ minimizes the above quantity and gives

$$E(\alpha,X) \ll (TX)^{\frac{3}{4}}\log^{\frac{5}{2}}X.$$

In particular, this gives an asymptotic formula for $M_{1,3}(X,\alpha)$ if $\Im(\alpha) \ll X^{\frac{1}{3}-\varepsilon}$.

Applying Lemma 5.4.6 to $E_0(X,\alpha)$ and Lemma 5.4.7 to $E_1(X,\alpha)$ we get Theorem 5.1.1 with an error

$$E(\alpha,X) \ll Y^{\frac{1}{2}}T^{\frac{1}{4}}X^{\frac{3}{4}+\varepsilon} + T^{\frac{3}{8}}Y^{-\frac{3}{4}}X^{\frac{7}{8}+\varepsilon} + T^{\frac{1}{2}}Y^{-\frac{2}{3}}X^{\frac{5}{6}+\varepsilon} + T^{\frac{1}{2}}X^{\frac{1}{2}+\varepsilon}.$$

If $T \leq X^{\frac{1}{4}}$, this is minimized by taking $Y = (XT)^{\frac{1}{10}}$ and in this range we have

$$E(\alpha, X) \ll T^{\frac{3}{10}} X^{\frac{4}{5}+\varepsilon}.$$

If $T \geq X^{\frac{1}{4}}$, we take $Y = X^{\frac{1}{14}} T^{\frac{3}{14}}$ and we get

$$E(\alpha, X) \ll T^{\frac{5}{14}} X^{\frac{11}{14}+\varepsilon}.$$

This concludes the proof of Theorem 5.1.1.

Part B

Period functions and cotangent sums

Chapter 6

Introduction

In the well-known theory of period polynomials one constructs a vector space of polynomials associated with a vector space of modular forms. The Hecke operators act on each space and have the same eigenvalues. Thus, either vector space produces the usual degree 2 L-series associated with holomorphic modular forms. In 2001 Lewis and Zagier extended this theory and defined spaces of period functions associated to non-holomorphic modular forms, i.e. to Maass forms and real analytic Eisenstein series. These period functions are real analytic functions $\psi(x)$ which satisfy three term relations

$$\psi(x) = \psi(x+1) + (x+1)^{-2s}\psi\left(\frac{x}{1+x}\right) \tag{6.0.1}$$

for some $s = \frac{1}{2} + it$. The period functions for Maass forms are characterized by (6.0.1) together with the growth conditions $\psi(x) = o(1/x)$ as $x \to 0^+$ and $\psi(x) = o(1)$ as $x \to \infty$; for these $s(1-s)$ is the Laplacian eigenvalue of the associated Maass form. The period functions for the Eisenstein series are the new solutions that one obtains by relaxing the above growth conditions replacing the o's by O's (by $O\left(\frac{1}{x|\log x|}\right)$ and $O(\log x)$ if $t = 0$).

Lewis and Zagier showed that the period functions ψ, which are initially defined only in the upper half plane, actually have an analytic continuation to all of $\mathbb{C}$ apart from the negative real axis. Moreover, each period function is associated to a periodic and holomorphic function f on the upper half plane,

$$f(z) = \psi(z) + z^{-2s}\psi\left(-\frac{1}{z}\right).$$

In this Part we use the work of Lewis and Zagier as a starting point, focusing in particular on the case of real analytic Eisenstein series. For these, the periodic function f turns out to be essentially

$$\sum_{n=1}^{\infty} \sigma_{2s-1}(n)\,\mathrm{e}(nz),$$

where, as usual, $\sigma_a(n) := \sum_{d|n} d^a$ indicates the sum of the a-th power of the divisors of n and $\mathrm{e}(z) := e^{2\pi iz}$. We interpret Lewis and Zagier's results directly in terms of this function and, in particular, in Chapter 7 we obtain a better understanding of the coefficients of the Taylor series of the associated period function. It turns out that the case $s = 1/2$ is especially useful. In this case the arithmetic part of the n-th Fourier coefficient is $d(n)$, the number of divisors of n, and one can show that the associated period function,

$$\sum_{n=1}^{\infty} d(n)e(nz) - \frac{1}{z}\sum_{n=1}^{\infty} d(n)e(-n/z),$$

which apparently only makes sense when the imaginary part of z is positive, actually has an analytic continuation to the split complex plane $\mathbb{C}'$ (the complex plane with the negative real axis removed). In the remaining chapters of this Part we will investigate some consequences of this surprising fact.

In Chapter 8 we give a generalization of the classical Voronoi summation formula, which is a formula for $\sum_{n=1}^{\infty} d(n)f(n)$ where $f(n)$ is a smooth rapidly decaying function. The usual formula proceeds from

$$\sum_{n=1}^{\infty} d(n)f(n) = \frac{1}{2\pi i}\int_{(2)} \zeta(s)^2 \tilde{f}(s)\ \mathrm{d}s$$

where

$$\tilde{f}(s) = \int_0^{\infty} f(x)x^{-s}\ \mathrm{d}x.$$

One obtains the formula by moving the path of integration to the left to $\Re(s) = -1$, say, and then using the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s)$$

of $\zeta(s)$, where, as usual, $\chi(s) := 2(2\pi)^{s-1}\Gamma(1-s)$. In this way one obtains a leading term

$$\int_0^\infty f(u)(\log u + 2\gamma)\,\mathrm{d}u,$$

from the pole of $\zeta(s)$ at $s = 1$, plus another term

$$\sum_{n=1}^\infty d(n)\hat{f}(n)$$

where $\hat{f}(u)$ is a kind of Fourier-Bessel transform of f; specifically,

$$\hat{f}(u) = \frac{1}{2\pi i}\int_{(-1)} \chi(s)^2 u^{s-1}\tilde{f}(s)\,\mathrm{d}s = \int_0^\infty f(t)C(2\pi\sqrt{tu})\,\mathrm{d}t$$

with $C(z) = 4K_0(2z) - 2\pi Y_0(2z)$ where K and Y are the usual Bessel functions. By contrast, we show that the period relation implies, for example, that for $0 < \delta < \pi$ and $z = 1 - e^{-i\delta}$

$$\sum_{n=1}^\infty d(n)\,\mathrm{e}(nz) = \frac{1}{4} + 2\frac{\log(-2\pi iz) - \gamma}{2\pi iz} + \frac{1}{z}\sum_{n=1}^\infty d(n)\,\mathrm{e}\left(\frac{-n}{z}\right) + \sum_{n=1}^\infty c_n e^{-in\delta} \tag{6.0.2}$$

where $\mathrm{c}_n \ll e^{-2\sqrt{\pi n}}$. This is a useful formula which cannot be readily extracted from the Voronoi formula. In fact, the Voronoi formula is actually an easy consequence of the formula (6.0.2). In Chapter 8 we give also some other applications of this extended Voronoi formula.

In Chapter 9 we give a second application, proving a surprising reciprocity formula for the Vasyunin sum, which is a cotangent sum that appears in the Nyman-Beurling criterion for the Riemann Hypothesis. Specifically, the Vasyunin sum appears as part of the exact formula for the twisted mean-square of the Riemann zeta-function on the critical line:

$$\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left(\frac{h}{k}\right)^{it} \frac{\mathrm{d}t}{\frac{1}{4} + t^2}.$$

The fact that there is a reciprocity formula for the Vasyunin sum is a non-obvious symmetry relating this integral for h/k and the integral for $\overline{h}/k$ where $h\overline{h} \equiv 1 \bmod k$. It is not apparent from this integral that there should be such a relationship; our formula reveals a hidden structure.

The reciprocity formula is most simply stated in terms of the function

$$\mathrm{c}_0\left(\frac{h}{k}\right) = -\sum_{m=1}^{k-1} \frac{m}{k} \cot \frac{\pi m h}{k} \tag{6.0.3}$$

defined initially for non-zero rational numbers h/k where h and k are integers with $(h,k) = 1$ and $k > 0$. The reciprocity formula can be simply stated as, “The function

$$\mathrm{c}_0\left(\frac{h}{k}\right) + \frac{k}{h}\,\mathrm{c}_0\left(\frac{k}{h}\right) - \frac{1}{\pi h}$$

extends from its initial definition on rationals $x = h/k$ to an (explicit) analytic function on the complex plane with the negative real axis deleted.” This is nearly an example of what Zagier calls a “quantum modular form.”

The reciprocity formula (6.0.3) can be extended to a whole family of cotangent sums c_a, which include as a particular case the Dedekind sum

$$\mathrm{s}\left(\frac{h}{k}\right) := -\frac{1}{4k}\sum_{m=1}^{k} \cot\left(\frac{\pi m}{k}\right)\cot\left(\frac{\pi m h}{k}\right).$$

In this case the resulting reciprocity formula is well known and has been generalized by Rademacher who showed

$$s\left(\frac{a\overline{b}}{c}\right) + s\left(\frac{b\overline{c}}{a}\right) + s\left(\frac{c\overline{a}}{b}\right) = \frac{a^2+b^2+c^2}{12abc} - \frac{1}{4},$$

for $(a,b) = (b,c) = (a,c) = 1$, $a, b, c \in \mathbb{N}^+$. In Chapter 10 we will provide the analogue of this result for all the cotangent sums c_a.

In the last two chapters we will give two applications of our study of period functions to averages of the Riemann zeta-function. In Chapter 11, we obtain a new formula for the weighted mean square of the Riemann zeta-function on the critical line,

$$\int_0^\infty |\zeta(1/2+it)|^2 e^{-\delta t}\ \mathrm{d}t.$$

Previously, the best formula for this quantity was a main term plus an asymptotic, but not convergent, series of powers of δ, each term an order of magnitude better than the previous as $\delta \to 0^+$. Our formula gives an asymptotic series which is also convergent.

In Chapter 12 we use the same technique to prove an exact formula for the second moment of the Riemann zeta function times a Dirichlet polynomial

$$\int_0^\infty \left| \zeta\left(\frac{1}{2} + it\right) \sum_{m \leq M} \frac{a_m}{m^{\frac{1}{2}+it}} \right|^2 e^{-\delta t} \, \mathrm{d}t.$$

Results on these integrals are particular useful, for example, when computing lower bounds for the proportion of zeros of ζ that lie on the critical line.

Chapter 7

Period functions

The work presented in this chapter is joint with J.B. Conrey and was first published in [BC1] and [BC2].

7.1 Introduction

For $a \in \mathbb{C}$ and $\Im(z) > 0$, consider

$$\mathcal{S}_a(z) := \sum_{n=1}^{\infty} \sigma_a(n)\, \mathrm{e}(nz), \tag{7.1.1}$$

where, as usual, $\sigma_a(n) := \sum_{d|n} d^a$. For $a = 2k+1$, $k \in \mathbb{Z}_{\geq 1}$, $\mathcal{S}_a(z)$ is essentially the Eisenstein series of weight $2k+2$,

$$E_{a+1}(z) = 1 + \frac{2}{\zeta(-a)} \mathcal{S}_a(z),$$

for which the well known modularity property

$$E_{2k}(z) - \frac{1}{z^{2k}} E_{2k}\left(-\frac{1}{z}\right) = 0$$

holds when $k \geq 2$. For other values of a this equality is no longer true, but the period function

$$\psi_a(z) := E_{a+1}(z) - \frac{1}{z^{a+1}} E_{a+1}\left(-\frac{1}{z}\right) \tag{7.1.2}$$

still has some remarkable properties.

Theorem 7.1.1. *Let* $\Im(z) > 0$ *and* $a \in \mathbb{C}$*. Then* $\psi_a(z)$ *satisfies the three term relation*

$$\psi_a(z) - \psi_a(z+1) = \frac{1}{(z+1)^{1+a}} \psi_a\left(\frac{z}{z+1}\right) \tag{7.1.3}$$

and extends to an analytic function on $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ *via the representation*

$$\psi_a(z) = \frac{i}{\pi z} \frac{\zeta(1-a)}{\zeta(-a)} - i \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} + i \frac{g_a(z)}{\zeta(-a)}, \tag{7.1.4}$$

where

$$\begin{aligned} g_a(z) :=& -2 \sum_{1 \leq n \leq M} (-1)^n \frac{B_{2n}}{(2n)!} \zeta(1-2n-a)(2\pi z)^{2n-1} + \\ &+ \frac{1}{\pi i} \int_{\left(-\frac{1}{2}-2M\right)} \zeta(s)\zeta(s-a)\Gamma(s) \frac{\cos\frac{\pi a}{2}}{\sin\frac{\pi(s-a)}{2}} (2\pi z)^{-s}\, ds, \end{aligned} \tag{7.1.5}$$

with B_n *denoting the* n*-th Bernoulli number, and* M *is any integer greater than or equal to* $-\frac{1}{2}\min(0, \Re(a))$*.*

Here and throughout the chapter equalities are to be interpreted as identities between meromorphic functions in a. In particular, taking the limit $a \to 0^+$, we have

$$\begin{aligned} \psi_0(z) &= -2 \frac{\log 2\pi z - \gamma}{\pi i z} - 2i g_0(z), \\ g_0(z) &= \frac{1}{\pi i} \int_{\left(-\frac{1}{2}\right)} \zeta(s)^2 \frac{\Gamma(s)}{\sin\frac{\pi s}{2}} (2\pi z)^{-s}\, ds = \frac{1}{\pi i} \int_{\left(-\frac{1}{2}\right)} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} z^{-s}\, ds. \end{aligned} \tag{7.1.6}$$

Theorem 7.1.1 is essentially a reformulation of Lewis and Zagier's results for the noncuspidal case in [LZ] and can be seen as a starting point for their theory of period functions.

For ease of reference, note that, in view of (7.1.4), (7.1.2) can be rewritten in terms of $\mathcal{S}_a$ and g_a as

$$\begin{aligned} &\mathcal{S}_a(z) - \frac{1}{z^{a+1}} \mathcal{S}_a\left(-\frac{1}{z}\right) = \\ &= i \frac{\zeta(1-a)}{2\pi z} - \frac{\zeta(-a)}{2} + \frac{e^{\frac{\pi i(a+1)}{2}} \zeta(a+1)\Gamma(a+1)}{(2\pi z)^{a+1}} + \frac{i}{2} g_a(z). \end{aligned} \tag{7.1.7}$$

Another important feature of the function $\psi_a(z)$ comes from the properties of its Taylor series. For example, in the case $a = 0$ one has

$$\frac{\pi i}{2}(1+z)\psi_0(1+z) = -1 - \frac{z}{2} + \sum_{m=2}^{\infty} a_m(-z)^m,$$

with

$$\begin{aligned} a_m &:= \frac{1}{n(n+1)} + 2b_n + 2\sum_{j=0}^{n-2} \binom{n-1}{j} b_{j+2}, \\ b_n &:= \frac{\zeta(n)B_n}{n} \end{aligned}$$

and where B_{2n} denotes the $2n$-th Bernoulli number. In particular, the values a_m are rational polynomials in π^2. The terms involved in the definition of a_m are extremely large, since

$$b_{2n} \sim \frac{B_{2n}}{2n} \sim (-1)^{n+1} 2\sqrt{\frac{\pi}{n}}\left(\frac{n}{\pi e}\right)^{2n}$$

as $n \to \infty$, though there is a lot of cancellation; for example, for $m = 20$ one has

$$\begin{aligned} a_m = &\frac{1}{420} + \frac{\pi^2}{36} - \frac{19\,\pi^4}{600} + \frac{646\,\pi^6}{19845} - \frac{323\,\pi^8}{1500} + \frac{4199\,\pi^{10}}{343035} + \\ &- \frac{154226363\,\pi^{12}}{36569373750} + \frac{1292\,\pi^{14}}{1403325} - \frac{248571091\,\pi^{16}}{2170943775000} + \\ &+ \frac{1924313689\,\pi^{18}}{288905366499750} - \frac{30489001321\,\pi^{20}}{252669361772953125} \\ = &\,0.0499998087\ldots \end{aligned}$$

Notice how close this number is to $\frac{1}{20}$; this observation can be made for all m and in fact one has

$$a_m - \frac{1}{m} \sim 2^{\frac{5}{4}}\pi^{\frac{3}{4}} \frac{e^{-2\sqrt{\pi m}}}{m^{\frac{3}{4}}} \sin\left(2\sqrt{\pi m} + \frac{3}{8}\pi\right).$$

A generalization of this asymptotic holds for the Taylor series at any point τ in the half plane $\Re(\tau) > 0$ and for any $a \in \mathbb{C}$. We give a proof in the following theorem, using g_a instead of ψ_a to simplify slightly the resulting formulae.

Theorem 7.1.2. *Let $\Re(\tau) > 0$ and for $|z| < |\tau|$, let*

$$g_a(\tau + z) := \sum_{m=0}^{\infty} \frac{g_a^{(m)}(\tau)}{m!} z^m$$

be the Taylor series of $g_a(z)$ around τ. Then

$$\begin{aligned} \frac{g_a^{(m)}(1)}{m!} = & - \sum_{\substack{2n-1+k=m,\\ n,k\geq 1}} (-1)^{n+m} B_{2n} \zeta(1-2n-a) \frac{\Gamma(2n+a+k)}{\Gamma(2n+a)k!(2n)!} 2(2\pi)^{2n-1} + \\ & + (-1)^m \cot\frac{\pi a}{2} \zeta(-a) \frac{\Gamma(1+a+m)}{\Gamma(1+a)m!} + \\ & + (-1)^m \left(\frac{\Gamma(1+a+m)}{\Gamma(a)(m+1)!} - 1 \right) \frac{\zeta(1-a)}{\pi}, \end{aligned} \tag{7.1.8}$$

and in particular if $a \in \mathbb{Z}_{\leq 0}$, $(a, m) \neq (0, 0)$, then $\pi g_a^{(m)}(1)$ is a rational polynomial in π^2. Moreover,

$$\begin{aligned} \frac{g_a^{(m)}(\tau)}{m!} = & \cos\left(\frac{\pi a}{2}\right) \frac{2^{\frac{7}{4}-\frac{a}{2}}}{\pi^{\frac{3}{4}+\frac{a}{2}}} \frac{e^{-2\sqrt{\pi\tau m}}}{m^{\frac{1}{4}-\frac{a}{2}} \tau^{m+\frac{3}{4}+\frac{a}{2}}} \times \\ & \times \left(\cos\left(2\sqrt{\pi\tau m} - \frac{\pi}{8}(2a-1) + (\tau+m)\pi\right) + O\left(\frac{1}{\sqrt{m}}\right)\right), \end{aligned} \tag{7.1.9}$$

as $m \to \infty$.

7.2 Analytic continuation of the period function

In this section we give a short proof of Theorem 7.1.1, following closely the work of Lewis and Zagier.

Proof of Theorem 7.1.1. Firstly, observe that the three term relation (7.1.3) follows easily from the periodicity in z of $E(a, z)$.

$\mathcal{S}_a(z)$ can be written as

$$\begin{aligned}\mathcal{S}_a(z) &= \sum_{n=1}^{\infty}\sigma_a(n)\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}\Gamma(s)(-2\pi inz)^{-s}\,ds\\ &= \frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}\zeta(s)\zeta(s-a)\Gamma(s)e^{\frac{\pi is}{2}}(2\pi z)^{-s}\,ds \qquad (7.2.1)\\ &= \frac{1}{2\pi i}\int_{\left(-\frac{1}{2}-2M\right)}\zeta(s)\zeta(s-a)\Gamma(s)e^{\frac{\pi is}{2}}(2\pi z)^{-s}\,ds + r_{a,M}(z),\end{aligned}$$

where M is any integer greater than or equal to $-\frac{1}{2}\min(0,\Re(a))$ and

$$\begin{aligned}r_{a,M}(z) :=& -\frac{1}{2}\zeta(-a) + i\frac{\zeta(1-a)}{2\pi z} + i\frac{\zeta(1+a)\Gamma(1+a)e^{\frac{\pi ia}{2}}}{(2\pi z)^{1+a}} +\\ &-\sum_{1\le n\le M} i(-1)^n\frac{B_{2n}}{(2n)!}\zeta(1-2n-a)(2\pi z)^{2n-1}\end{aligned}$$

is the sum of the residues encountered moving the integral (and has to be interpreted in the limit sense if some of the terms have a pole). Now, consider

$$\begin{aligned}\frac{1}{z^{1+a}}\mathcal{S}_a\left(-\frac{1}{z}\right) &= \frac{1}{z^{1+a}}\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}\zeta(s)\zeta(s-a)\Gamma(s)e^{\frac{\pi is}{2}}\left(2\pi\frac{-1}{z}\right)^{-s}ds\\ &= \frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}\zeta(s)\zeta(s-a)\Gamma(s)e^{-\frac{\pi is}{2}}(2\pi)^{-s}z^{s-1-a}\,ds,\end{aligned}$$

since in this context $0 < \arg z < \pi$ and $0 < \arg\frac{-1}{z} < \pi$, so the identity $\arg\frac{-1}{z} = \pi - \arg z$ holds. Applying the functional equation to both $\zeta(s)$ and $\zeta(s-a)$ we get, after the change of variable $s \to 1-s+a$,

$$\begin{aligned}\frac{1}{z^{1+a}}\mathcal{S}_a\left(-\frac{1}{z}\right) &= -\frac{1}{2\pi}\int_{(-1+\min(0,\Re(a)))}\zeta(s-a)\zeta(s)\Gamma(s)\frac{e^{\frac{\pi i(s-a)}{2}}\cos\frac{\pi s}{2}}{\sin\frac{\pi(s-a)}{2}}(2\pi z)^{-s}\,ds\\ &= -\frac{1}{2\pi}\int_{\left(-\frac{1}{2}-M\right)}\zeta(s-a)\zeta(s)\Gamma(s)\frac{e^{\frac{\pi i(s-a)}{2}}\cos\frac{\pi s}{2}}{\sin\frac{\pi(s-a)}{2}}(2\pi z)^{-s}\,ds,\end{aligned} \qquad (7.2.2)$$

since the integrand doesn't have any pole on the left of $-1+\min(0,\Re(a))$. The theorem then follows summing (7.2.1) and (7.2.2) and using the identity

$$e^{\frac{\pi is}{2}} + i\frac{e^{\frac{\pi i(s-a)}{2}}\cos\frac{\pi s}{2}}{\sin\frac{\pi(s-a)}{2}} = i\frac{\cos\frac{\pi a}{2}}{\sin\frac{\pi(s-a)}{2}}.$$

$\square$

We remark that for $a = 2k + 1$, $k \geq 1$, Theorem 7.1.1 reduces to

$$E_{2k}(z) - \frac{1}{z^{2k}} E_{2k}\left(-\frac{1}{z}\right) = 0,$$

while, for $a = 1$, the theorem reduces to the well known identity

$$E_2(z) - E_2\left(\frac{-1}{z}\right) = -\frac{12}{2\pi i z}.$$

7.3 The Taylor coefficients

In this section we give a proof of Theorem 7.1.2. We start with the following lemma.

Lemma 7.3.1. *For fixed complex numbers A and α we have, as $n \to \infty$*

$$J_n := \int_0^\infty u^{n+\alpha} e^{-A\sqrt{u}} e^{-u} \frac{\mathrm{d}u}{u} = \sqrt{2\pi} e^{\frac{A^2}{8}} e^{-A\sqrt{n}} e^{-n} n^{n+\alpha-\frac{1}{2}} \left(1 - \frac{C}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right),$$

where

$$C = \frac{4\alpha - 1}{8} A + \frac{A^3}{96}.$$

Proof. After the change of variable $u = nx^2$ we have

$$\begin{aligned} J_n &= 2n^{n+\alpha} \int_0^\infty x^{2\alpha-1} e^{-A\sqrt{n}x - n\left(x^2 - 2\log x\right)} \mathrm{d}x \\ &= 2n^{n+\alpha} e^{-A\sqrt{n}} \int_{-1}^\infty (x+1)^{2\alpha-1} e^{-A\sqrt{n}x - n\left((x+1)^2 - 2\log(x+1)\right)} \mathrm{d}x \\ &= 2n^{n+\alpha} e^{-A\sqrt{n}} e^{-n} \left(1 + O\left(e^{-\frac{n\delta^2}{2}}\right)\right) \times \\ &\quad \times \int_{-\delta}^{\delta} (x+1)^{2\alpha-1} e^{-A\sqrt{n}x - 2nx^2} \left(1 + \frac{2nx^3}{3} + O\left(nx^4\right)\right) \mathrm{d}x, \end{aligned}$$

for any small $\delta > 0$. We can then approximate the binomial and extend the integral to $\mathbb{R}$ at a negligible cost, getting

$$\begin{aligned} J_n = 2n^{n+\alpha} e^{-A\sqrt{n}} e^{-n} \int_{-\infty}^\infty \left(1 + (2\alpha - 1)x + \frac{2nx^3}{3} + O\left(x^2 + nx^4\right)\right) \times \\ \times e^{-A\sqrt{n}x - 2nx^2} \mathrm{d}x. \end{aligned}$$

Evaluating the integrals, the lemma follows. □

Proof of Theorem 7.1.2. The three term relation (7.1.3) implies that

$$\begin{aligned} g_a(z+1) =& \frac{1}{(z+1)^{1+a}}\cot\frac{\pi a}{2}\zeta(-a) - \frac{1}{\pi z(z+1)^a}\zeta(1-a) + \\ &+ \frac{1}{\pi z(z+1)}\zeta(1-a) + g_a(z) - \frac{1}{(z+1)^{1+a}} g_a\left(\frac{z}{z+1}\right). \end{aligned}$$

Now, writing $c_n(a) := \frac{\zeta(1-2n-a)B_{2n}}{2n!}$, from the definition (7.1.5) of $g_a(z)$, it follows that

$$g_a(z) = 2\sum_{1\le n\le M}(-1)^n c_n(a)(2\pi z)^{2n-1} + O\left(|z|^{2M+\frac{1}{2}}\right),$$

for any $M \ge 1$. Thus

$$\begin{aligned} & g_a(z) - \frac{g_a\left(\frac{z}{z+1}\right)}{(z+1)^{1+a}} = \\ &= 2\sum_{1\le n\le M}(-1)^n c_n(a)(2\pi z)^{2n-1}\left(1-\frac{1}{(z+1)^{2n+a}}\right) + O\left(|z|^{2M+\frac{1}{2}}\right) \\ &= -2\sum_{m=1}^{2M}\left(\sum_{\substack{2n-1+k=m,\\ n,k\ge 1}}(-1)^{n+m}c_n(a)\frac{\Gamma(2n+a+k)}{\Gamma(2n+a)k!}(2\pi)^{2n-1}\right)z^m + O\left(|z|^{2M+\frac{1}{2}}\right). \end{aligned}$$

Therefore,

$$g_a(z+1) = \sum_{m=0}^{2M} b_m z^m + O(|z|^{2M+\frac{1}{2}}),$$

where

$$\begin{aligned} b_m :=& -2\sum_{\substack{2n-1+k=m,\\ n,k\ge 1}}(-1)^{n+k}B_{2n}\zeta(1-2n-a)\frac{\Gamma(2n+a+k)}{\Gamma(2n+a)k!(2n)!}(2\pi)^{2n-1} + \\ &+(-1)^m\cot\frac{\pi a}{2}\zeta(-a)\frac{\Gamma(1+a+m)}{\Gamma(1+a)m!} + \\ &+(-1)^m\left(\frac{\Gamma(1+a+m)}{\Gamma(a)(m+1)!}-1\right)\frac{\zeta(1-a)}{\pi}, \end{aligned}$$

and, since $g_a(z)$ is holomorphic at 1, b_m must coincide with the m-th coefficient of the Taylor series of $g_a(z)$ at 1.

Now, let's prove the asymptotic (7.1.9). Fix any $M \ge -\frac{1}{2}\min(0, \Re(a))$ and assume $\Re(\tau) > 0$. By the functional equation for ζ and basic properties of

$\Gamma(s)$, we have

$$\begin{aligned}
&\frac{(2\pi)^a\tau^m}{\cos\frac{\pi a}{2}}g_a^{(m)}(\tau) = \\
&= \frac{(-1)^m}{\pi i}\int_{\left(-\frac{1}{2}-2M\right)}\Gamma(s)\frac{\zeta(s)\zeta(s-a)}{\sin\frac{\pi(s-a)}{2}}s(s+1)\cdots(s+m-1)(2\pi)^{-s+a}\tau^{-s}\,ds \\
&= \frac{(-1)^m}{\pi i}\int_{\left(-\frac{1}{2}-2M\right)}\frac{\zeta(s)\zeta(s-a)}{\sin\frac{\pi(s-a)}{2}}\Gamma(s+m)(2\pi)^{-s+a}\tau^{-s}\,ds \\
&= \frac{(-1)^m}{\pi^3 i}\int_{\left(-\frac{1}{2}-2M\right)}\zeta(1-s)\zeta(1-s+a)\times \\
&\qquad\qquad \times\Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m)\sin\frac{\pi s}{2}\left(\frac{2\pi}{\tau}\right)^s ds.
\end{aligned}$$

We can see immediately that $g_a^{(m)}(\tau) \ll_a m^{-B}|\tau|^{-m}m!$ for any fixed $B > 0$, just by moving the path of integration to the line $\Re(s) = -B$ and using trivial estimates for Γ. To get a formula which is asymptotic as $m \to \infty$ we expand $\zeta(1-s)\zeta(1-s+a)$ into a Dirichlet series and integrate term-by-term; the main term arises from the first term of the sum. We have

$$g_a^{(m)}(\tau) = 2\frac{(-\tau)^m\cos\frac{\pi a}{2}}{\pi^2(2\pi)^a}\sum_{\ell=1}^{\infty}\frac{\sigma_{-a}(\ell)}{\ell}\Upsilon_{m,a}\left(\frac{\ell}{\tau}\right), \tag{7.3.1}$$

where

$$\Upsilon_{m,a}(x) := \frac{1}{2\pi i}\int_{\left(-\frac{1}{2}-2M\right)}\Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m)\sin\frac{\pi s}{2}(2\pi x)^s\,ds.$$

By Euler's formula, we can decompose $\Upsilon_{m,a}(x)$ into

$$\Upsilon_{m,a}(x) = \frac{\Upsilon_{m,a}^{+}(x) - \Upsilon_{m,a}^{-}(x)}{2i}, \tag{7.3.2}$$

where

$$\Upsilon_{m,a}^{\pm}(x) := \frac{1}{2\pi i}\int_{\left(-\frac{1}{2}-2M\right)}\Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m)(\pm 2\pi i x)^s\,ds. \tag{7.3.3}$$

We re-express these integrals as convolution integrals. Recall that for $|\arg x| < \pi$ we have

$$\frac{1}{2\pi i}\int_{\left(\frac{3}{2}+2M\right)}\Gamma(s)\Gamma(s+a)u^{-s}\,ds = 2u^{\frac{a}{2}}K_a\big(2\sqrt{u}\big),$$

where K_a denotes the K-Bessel function of order a. Also, if $m \geq 2M+1$,

$$\frac{1}{2\pi i}\int_{\left(-\frac{1}{2}-2M\right)} \Gamma(s+m)u^{-s}\,ds = u^m e^{-u}.$$

Thus, for $m \geq 2M+1$, we have

$$\Upsilon^{\pm}_{m,a}(x) = \pm 2i(2\pi x)^{1+\frac{a}{2}} e^{\frac{\pm\pi i a}{4}} \int_0^{\infty} u^{m+\frac{a}{2}} K_a\Big(2e^{\pm\frac{\pi i}{4}}\sqrt{2\pi x u}\Big) e^{-u}\,\mathrm{d}u.$$

Now, for $|\arg z| < \frac{3}{2}\pi$

$$K_a(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\bigg(1+\frac{4a^2-1}{8z}+O_a\Big(\frac{1}{|z|^2}\Big)\bigg),$$

as $z \to \infty$, and

$$K_{-a}(z) = K_a(z) \sim \begin{cases} 2^{a-1}\Gamma(a)z^{-a}, & \text{if } \Re(a) \geq 0,\ a \neq 0, \\ -\log\frac{x}{2}-\gamma, & \text{if } a = 0, \end{cases}$$

as $z \to 0$ (see Sections 7.23 and 3.7-3.71 in [Wats]). Therefore,

$$\begin{aligned}\Upsilon^{\pm}_{m,a}(x) = {} & \pm 2i(2\pi x)^{1+\frac{a}{2}} \frac{\pi^{\frac{1}{4}} e^{\frac{\pm\pi i\left(a-\frac{1}{2}\right)}{4}}}{2^{\frac{5}{4}}x^{\frac{1}{4}}} \int_0^{\infty} u^{m+\frac{a}{2}-\frac{1}{4}} e^{-u-2(1\pm i)\sqrt{\pi x u}} \times \\ & \times \left(1+\frac{4a^2-1}{2^{\frac{9}{2}}\pi^{\frac{1}{2}}e^{\pm\frac{\pi i}{4}}\sqrt{xu}}+O_a\Big(\frac{1}{u|x|}\Big)\right)\mathrm{d}u.\end{aligned}$$

Thus, by Lemma 7.3.1,

$$\begin{aligned}\frac{\Upsilon^{\pm}_{m,a}(x)}{2i} \sim {} & \pm 2^{\frac{1}{4}+\frac{a}{2}}\pi^{\frac{7}{4}+\frac{a}{2}} e^{\frac{\pm\pi i\left(a-\frac{1}{2}\right)}{4}} x^{\frac{3}{4}+\frac{a}{2}} e^{\pm i\pi x} e^{-2(1\pm i)\sqrt{\pi x m}} e^{-m} m^{m+\frac{1}{4}+\frac{a}{2}} \times \\ & \times \left(1+\frac{\xi^{\pm}}{\sqrt{m}}+O\Big(\frac{1}{m|x|}\Big)\right),\end{aligned} \tag{7.3.4}$$

uniformly in $|x| \ll 1$, where

$$\xi^{\pm} = -\frac{(1\pm i)\sqrt{\pi x}(1+a)}{2}+\frac{(1\mp i)(\pi x)^{\frac{3}{2}}}{6}+\frac{(4a^2-1)(1\mp i)}{32\pi^{\frac{1}{2}}\sqrt{x}}.$$

Thus, (7.1.9) follows by (7.3.1) and (7.3.4). (Notice that, for $\ell \geq 2$ and $\Re(\tau) > 0$, we have $\Upsilon^{\pm}_{m,a}\big(\frac{\ell}{\tau}\big) \ll \Upsilon^{\pm}_{m,\Re(a)}\big(\Re\big(\frac{2}{\tau}\big)\big)$ and so we don't have to worry about the lack of uniformity in (7.3.4) for x large.) $\square$

Chapter 8

An extension of Voronoi's summation formula

The work presented in this chapter is joint with J.B. Conrey and was first published in [BC2].

8.1 Introduction

At the beginning of the 20th century, in an attempt to study sums of the divisor function $d(n)$, G.F. Voronoi proved a summation formula, which, in its simplest form, states that, if $f(u)$ is a smooth function of compact support, then

$$\sum_{n=1}^{\infty} d(n) f(n) = \sum_{n=1}^{\infty} d(n) \hat{f}(n) + \int_0^{\infty} f(t)(\log t + 2\gamma)\, \mathrm{d}t + \frac{f(0)}{4}, \tag{8.1.1}$$

where

$$\hat{f}(x) := 4 \int_0^{\infty} f(t) \Big(K_0\Big(4\pi\sqrt{tx}\Big) - \frac{\pi}{2} Y_0\Big(4\pi\sqrt{tx}\Big)\Big)\, \mathrm{d}t.$$

This formula later became a fundamental tool in analytic number theory and the study of its extension to other settings has become a very active area of research.

In this chapter we show that the ideas used in the proofs of Theorem 7.1.1 and 7.1.2 can be easily generalized providing an extension of Voronoi's summation formula. For example, let $F(s)$ be a meromorphic function on $1-\omega \leq \Re(s) \leq \omega$ for some $1 < \omega < 2$ with no poles on the boundary and assume $|F(\sigma+it)| \ll_\sigma e^{\left(\frac{\pi}{2}-\eta\right)|t|}$ for some $\eta > 0$. Let

$$\begin{aligned} W_+(z) &:= \frac{1}{2\pi i}\int_{(\omega)} F(s)\Gamma(s)(-2\pi i z)^{-s}\,ds, \\ W_-(z) &:= \frac{1}{2\pi i}\int_{(\omega)} F(1-s)\Gamma(s)(-2\pi i z)^{-s}\,ds, \end{aligned} \tag{8.1.2}$$

for $\frac{\pi}{2}-\eta < \arg z < \frac{\pi}{2}+\eta$. (Notice that these functions are essentially convolutions of the exponential function and the Mellin transform of $F(s)$.) Then we have

$$\sum_{n=1}^{\infty} d(n)W_+(nz) - \frac{1}{z}\sum_{n=1}^{\infty} d(n)W_-\Big(-\frac{n}{z}\Big) = R(z) + k(z), \tag{8.1.3}$$

where $R(z)$ is the sum of the residues of $F(s)\Gamma(s)\zeta(s)^2(-2\pi i z)^{-s}$ between $1-\omega$ and ω, and

$$k(z) := \frac{1}{2\pi}\int_{(1-\omega)} F(s)\frac{\zeta(s)\zeta(1-s)}{\sin \pi s} z^{-s}\,ds$$

is holomorphic on $|\arg(z)| < \frac{\pi}{2}+\eta$. Moreover, if we assume that $F(s)$ is holomorphic in $\Re(s) < 1-\omega$, then it follows that the Taylor series of $k(z)$ converges very fast,

$$\frac{k^{(n)}(\tau)}{n!} \ll n^{-B}|\tau|^{-n}$$

for any $B > 0$ and τ such that $|\arg \tau| < \eta$ (and where the implied constant may depend on B and η). Also, $W_-(z)$ decays faster than any power of z at infinity and so the second sum in the left hand side of (8.1.3) is rapidly convergent and is very small if we let z go to zero in $|\arg z| < \eta$. In Section 8.2 we will give an explicit example.

In Section 8.3 we will show that the Voronoi summation formula can be deduced from (8.1.3) (or also directly from (7.1.7)) as a very easy corollary.

In particular, Voronoi's formula can be interpreted as a version of the formula (7.1.7) confined to the positive real axis. If we get rid of this limitation and we use directly the period formula (7.1.7), we are able to obtain interesting results also for weight functions of the shape $f(u) = e^{-\delta u}$, for which the Voronoi summation formula fails to give a useful formula.

The use of a weight function of the shape $e^{-\delta u}$ is fundamental to study the smoothly weighted second moment of the Riemann zeta function,

$$\int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 e^{-\delta t}\,\mathrm{d}t.$$

We will investigate this problem in Chapter 11.

8.2 An extension of Voronoi's formula

Formula (8.1.3) can be proved in exactly the same way as in the proof of Theorems 7.1.1 and 7.1.2 (with $a = 0$). In this section we give an application of this formula and we discuss a similar formula for convolutions of the exponential function.

Applying formula (8.1.3) to $F(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma(s)}$ we get, for $\frac{\pi}{4} < \arg(z) < \frac{3}{4}\pi$,

$$\sum_{n=1}^\infty d(n) e^{(2\pi n z)^2} = \frac{1}{z}\sum_{n=1}^\infty d(n) T(4\pi n z) + R(z) + k(z), \tag{8.2.1}$$

where, for $\frac{\pi}{4} < \arg(z) < \frac{3}{4}\pi$,

$$T(z) := \frac{1}{\sqrt{\pi} i}\int_{(2)} \frac{\Gamma(s)}{\Gamma\left(1-\frac{s}{2}\right)}(-iz)^{-s}\,ds = \sum_{n=0}^\infty \frac{(iz)^n}{n!\Gamma\left(1+\frac{n}{2}\right)}$$

and

$$\begin{aligned} R(z) &:= = \frac{1}{4} + \frac{2\log(-4\pi i z) - 3\gamma}{8\sqrt{\pi} i z}, \\ k(z) &:= \frac{1}{4\pi^2}\int_{\left(-\frac{1}{2}\right)} \Gamma\left(\frac{s}{2}\right)\Gamma(1-s)\zeta(s)\zeta(1-s) z^{-s}\,ds. \end{aligned}$$

Notice that we have $T(z) \ll |z|^{-B}$ for all fixed $B > 0$; moreover, $k(z)$ is holomorphic in $|\arg(z)| < \frac{3}{4}\pi$ and, if $|\arg(\tau)| < \frac{\pi}{4}$, one has that the m-th coefficient of the Taylor series of $k(z)$ is very small. More precisely, one has

$$c_\tau(m) := \frac{k^{(m)}(\tau)}{m!} \ll |\tau|^{-m} m^{-B}$$

for all fixed $B > 0$. In particular, if we set $z = i\delta$ with $0 < \delta \le 1$, taking the real part of (8.2.1) we get

$$\sum_{n=1}^{\infty} d(n) e^{-(2\pi n\delta)^2} = \frac{1}{4} + \frac{-2\log(4\pi\delta) - 3\gamma}{4\sqrt{\pi}\delta} + \Re \sum_{m=0}^{\infty} c_m \left(\frac{\sqrt{3}}{2} + i\left(\frac{1}{2} - \delta\right)\right)^m \tag{8.2.2}$$

with

$$c_m := c_{\frac{\sqrt{3}+i}{2}}(m) \ll m^{-B}.$$

for all fixed $B > 0$.

We now state a similar formula for convolutions of the exponential function and a function that is compactly supported on $\mathbb{R}_{>0}$.

Let $g(x)$ be a compactly supported function on $\mathbb{R}_{>0}$ and let

$$W_+(z) := \int_0^\infty f\left(\frac{1}{x}\right) \mathrm{e}(zx) \frac{\mathrm{d}x}{x}$$
$$W_-(z) := \int_0^\infty f(x) \,\mathrm{e}(zx) \,\mathrm{d}x.$$

If we denote the Mellin transform of $f(x)$ with $F(s)$, then it follows that $F(s)$ is entire and that $W_+(x)$ and $W_-(x)$ can be written as in (8.1.2). In particular, since

$$F(0) = \int_0^\infty f(x) \frac{\mathrm{d}x}{x},$$
$$F(1) = \int_0^\infty f(x) \,\mathrm{d}x,$$
$$F'(1) = \int_0^\infty f(x) \log x \,\mathrm{d}x,$$

formula (8.1.3) can be written as

$$\begin{aligned}\sum_{n=1}^{\infty} d(n) W_+(nz) - \frac{1}{z}\sum_{n=1}^{\infty} d(n) W_-\Big(-\frac{n}{z}\Big) = \\ = \int_0^\infty f(x) \left(\frac{1}{4x} - \frac{1}{4z} - \frac{\gamma - \log(2\pi z/x)}{2\pi i z}\right) \mathrm{d}x + k(z) + \\ + \int_0^\infty f(x) \int_{(-\frac{1}{2})} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} \Big(\frac{z}{x}\Big)^{-s} ds \,\frac{\mathrm{d}x}{2\pi x},\end{aligned} \tag{8.2.3}$$

for $\Im(z) > 0$.

8.3 A short proof of Voronoi's formula

In this section we show how to use the results of this chapter to give a short proof of Voronoi's formula.

Proof of Voronoi's formula. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a smooth function that decays faster than any power of x and let

$$\tilde{f}(x) := 2\int_0^\infty f(y)\cos(2\pi xy)\,\mathrm{d}y$$

be the cosine transform of $f(x)$. Then, $\tilde{f}(x)$ is smooth and, by partial integration, $\tilde{f}^{(m)}(x) \ll \frac{1}{x^{2+m}}$ for all $m \geq 0$, where $g^{(m)}(x)$ denotes the m-th derivative of $g(x)$. For $0 < \Re(s) < 2$, we can define the Mellin transform of $\tilde{f}$,

$$F(s) := \int_0^\infty \tilde{f}(x)x^{s-1}\,\mathrm{d}x.$$

By partial integration we see that $F(s)$ extends to a meromorphic function on $\Re(s) < 2$ with simple poles at most at the non-positive integers. Also, $F(s)$ decays rapidly on vertical strips. Moreover, since for $0 < \Re(s) < 1$

$$2\int_0^\infty y^{s-1}\cos(2\pi xy)\,\mathrm{d}y = 2(2\pi x)^{-s}\Gamma(s)\cos\frac{\pi s}{2},$$

then, for $0 < \Re(s) < 1$, by Parseval's formula we have

$$\begin{aligned} F(s) &= \frac{2}{s}\int_0^\infty f(y)(2\pi y)^{-s}\Gamma(s+1)\cos\frac{\pi s}{2}\,\mathrm{d}y \\ &= \frac{2}{s}\int_0^\infty f(y)\,\mathrm{d}y - 2\int_0^\infty f(y)(\log(2\pi y)+\gamma)\,\mathrm{d}y + O(|s|) \\ &= \frac{F_{-1}}{s} + F_0 + O(|s|), \end{aligned}$$

say. For $\Im(z) \geq 0$ we can define

$$\begin{aligned} W_+(z) &:= \frac{1}{2\pi i}\int_{(\frac{3}{2})} F(s)\Gamma(s)(-2\pi iz)^{-s}\mathrm{d}s, \\ W_-(z) &:= \frac{1}{2\pi i}\int_{(\frac{3}{2})} F(1-s)\Gamma(s)(-2\pi iz)^{-s}\mathrm{d}s, \end{aligned} \tag{8.3.1}$$

or, by convolution,

$$\begin{aligned} W_+(z) &= \int_0^\infty \tilde{f}\left(\frac{1}{x}\right) \mathrm{e}(zx) \frac{\mathrm{d}x}{x}, \\ W_-(z) &= \int_0^\infty \left(\tilde{f}(x) - \mathrm{Res}_{s=0} F(s)\right) \mathrm{e}(zx)\, \mathrm{d}x, \end{aligned} \tag{8.3.2}$$

with the second representation of $W_-(z)$ defined only in $\Im(z) > 0$.

Since $F(s)$ is rapidly decaying at infinity, (8.1.3) holds for $\Im(z) \geq 0$ and so we can apply it to $z = 1$ and take the real part. By the definition of $\tilde{f}$, we have

$$\begin{aligned} \Re(W_+(n)) &= 2\int_0^\infty f(y) \int_0^\infty \cos\left(\frac{2\pi y}{x}\right) \cos(nx) \frac{\mathrm{d}x}{x}\, \mathrm{d}y \\ &= \int_0^\infty f(y)(2K_0(4\pi\sqrt{ny}) - \pi Y_0(4\pi\sqrt{ny}))\, \mathrm{d}y \end{aligned}$$

and

$$\begin{aligned} \Re(W_-(-n)) &= \lim_{\substack{z\to 1,\\ \Im(z)>0}} \Re(W_-(-nz)) \\ &= \lim_{\substack{z\to 1,\\ \Im(z)>0}} \Re \int_0^\infty \tilde{f}(x)\, \mathrm{e}(-nzx)\, \mathrm{d}x - \lim_{\substack{z\to 1,\\ \Im(z)>0}} \Re \frac{\mathrm{Res}_{s=0} F(s)}{-2\pi i n z} \\ &= \frac{1}{2} f(n), \end{aligned}$$

since $\mathrm{Res}_{s=0} F(s)$ is real. Moreover,

$$k(z) = \frac{1}{2\pi} \int_{(-\frac{1}{2})} F(s) \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} z^{-s}\, ds$$

is purely imaginary on the real line, so we just need to compute

$$\begin{aligned} \Re\left(\operatorname*{Res}_{s=0,1} F(s)\Gamma(s)\zeta(s)^2(-2\pi i)^{-s}\right) &= \\ =&\Re\left(\frac{F(1)(\gamma - \log(-2\pi i)) + F'(1)}{-2\pi i}\right. \\ &\left. + \frac{-F_{-1}(\log(-2\pi i) + \gamma - 2\log 2\pi) + F_0}{4}\right) \\ =& -\frac{f(0)}{8} - \frac{1}{2}\int_0^\infty f(y)(\log y + 2\gamma)\, \mathrm{d}y, \end{aligned}$$

since $F(1) = \frac{f(0)}{2}$ and $F'(1)$ is real. This completes the proof of the theorem.

$\square$

Chapter 9

Cotangent sums

The work presented in this chapter is joint with J.B. Conrey and was first published in [BC1] and [BC2].

9.1 Introduction

For a rational number $\frac{h}{k}$, $(h,k)=1$, $k>0$, define

$$\mathrm{c}_0\left(\frac{h}{k}\right) = -\sum_{m=1}^{k-1} \frac{m}{k} \cot\left(\frac{\pi m h}{k}\right).$$

The value of $\mathrm{c}_0\left(\frac{h}{k}\right)$ is an algebraic number, i.e. $c_0 : \mathbb{Q} \to \overline{\mathbb{Q}}$, and, more precisely, $i\,\mathrm{c}_0\left(\frac{h}{k}\right)$ is purely imaginary and is contained in the cyclotomic field of $2k$-th roots of unity. Moreover, c_0 is odd and is periodic of period 1.

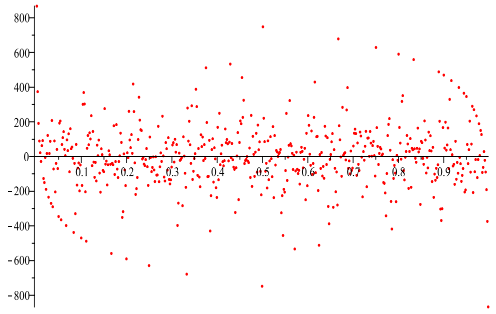

Figure 9.1: Graph of $\mathrm{c}_0\left(\frac{h}{k}\right)$ for $1 \leq h < k = 541$.

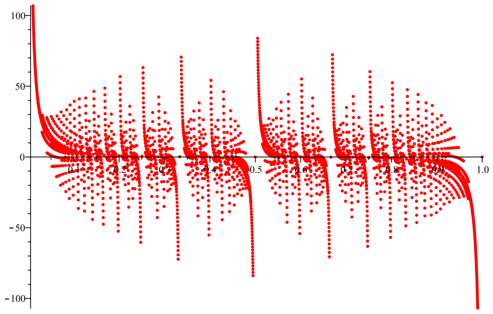

Figure 9.2: Graph of $\mathrm{c}_0\left(\frac{h}{k}\right)$ for $1 \leq h \leq k \leq 100$, $(h,k)=1$.

The cotangent sum $\mathrm{c}_0\big(\frac{h}{k}\big)$ arises in analytic number theory in the value at $s = 0$,

$$D\left(0, \frac{h}{k}\right) = \frac{1}{4} + \frac{i}{2}\,\mathrm{c}_0\left(\frac{h}{k}\right), \tag{9.1.1}$$

of the Estermann function, defined for $\Re(s) > 1$ by

$$D\left(s, \frac{h}{k}\right) := \sum_{n=1}^{\infty} \frac{d(n)\,\mathrm{e}(nh/k)}{n^s}.$$

The Estermann function extends analytically to $\mathbb{C} \setminus \{1\}$ and satisfies a functional equation; these properties are useful in studying the asymptotics of the mean square of the Riemann zeta function multiplied by a Dirichlet polynomial (see [BCH-B]), which are needed, for example, for theorems which give a lower bound for the portion of zeros of $\zeta(s)$ on the critical line. See also [Con] and [Iwa80]. The sum

$$V\left(\frac{h}{k}\right) := \sum_{m=1}^{k-1} \left\{\frac{mh}{k}\right\} \cot\left(\frac{\pi m}{k}\right) = -\,\mathrm{c}_0\left(\frac{\overline{h}}{k}\right),$$

known as the Vasyunin sum, arises in the study of the Riemann zeta function by virtue of the formula:

$$\begin{aligned}\nu\left(\frac{h}{k}\right) &:= \frac{1}{2\pi\sqrt{hk}} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left(\frac{h}{k}\right)^{it} \frac{\mathrm{d}t}{\frac{1}{4} + t^2} \\ &= \frac{\log 2\pi - \gamma}{2}\left(\frac{1}{h} + \frac{1}{k}\right) + \frac{k-h}{2hk}\log\frac{h}{k} - \frac{\pi}{2hk}\left(V\left(\frac{h}{k}\right) + V\left(\frac{k}{h}\right)\right).\end{aligned} \tag{9.1.2}$$

This formula is relevant to the Nyman-Beurling-Báez-Duarte approach to the Riemann hypothesis which asserts that the Riemann hypothesis is true if and only if $\lim_{N\to\infty} d_N = 0$, where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - \zeta A_N\left(\frac{1}{2} + it\right)\right|^2 \frac{\mathrm{d}t}{\frac{1}{4} + t^2}$$

and the inf is over all the Dirichlet polynomials $A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ of length N; see [Bag] for a nice account of the Nyman-Beurling approach to the Riemann hypothesis with Báez-Duarte's significant contribution and see [BBLS05]

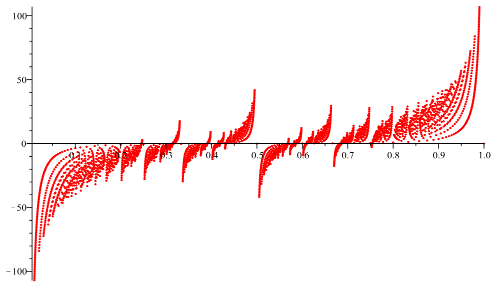

Figure 9.3: Graph of $V\left(\frac{h}{k}\right)$ for $1 \leq h, k \leq 100$ and $(h,k)=1$.

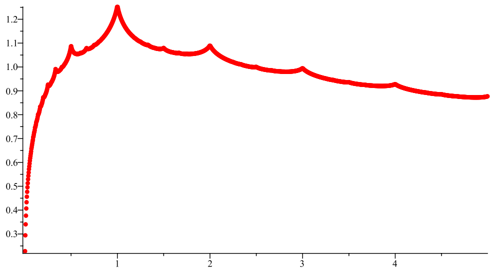

Figure 9.4: Graph of $\sqrt{hk}\,\nu\left(\frac{h}{k}\right)$ for $1 \leq h \leq 5k$, $k=307$, $(h,k)=1$.

and [LR] for information about the Vasyunin sums, as well as interesting numerical experiments about d_N and the minimizing polynomials A_N. Thus d_N^2 is a quadratic expression in the unknown quantities a_m in terms of the Vasyunin sums.

In this chapter we present a new reciprocity formula for $\mathrm{c}_0\left(\frac{h}{k}\right)$.

Theorem 9.1.1. *Let $(h,k)=1$, $h,k \geq 1$. Then,*

$$\mathrm{c}_0\left(\frac{h}{k}\right) + \frac{k}{h}\,\mathrm{c}_0\left(\frac{k}{h}\right) - \frac{1}{\pi h} = \frac{i}{2}\psi_0\left(\frac{h}{k}\right), \tag{9.1.3}$$

where $\psi_0(z)$ is given by (7.1.6)*. In particular $\mathrm{c}_0\left(\frac{h}{k}\right) + \frac{k}{h}\,\mathrm{c}_0\left(\frac{k}{h}\right) - \frac{1}{\pi h}$ is analytic on $\mathbb{C}'$.*

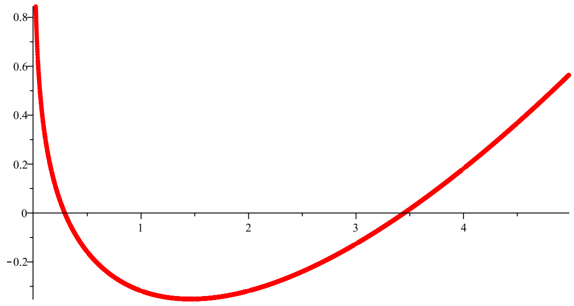

Figure 9.5: Graph of $\mathrm{c}_0\left(\frac{h}{k}\right)+\frac{k}{h}\,\mathrm{c}_0\left(\frac{k}{h}\right)-\frac{1}{\pi h}$ for $h \leq 5k$, $k \leq 50$ and $(h,k)=1$.

The cotangent sum $\mathrm{c}_0\left(\frac{h}{k}\right)$ appears to be nowhere continuous with respect to the real topology (as can be verified intuitively looking at Figures 9.1 and 9.2), but Theorem 9.1.1 shows that $\mathrm{c}_0\left(\frac{h}{k}\right) + \frac{k}{h}\,\mathrm{c}_0\left(\frac{k}{h}\right)$ extends to an analytic function, once we subtract the correction term $\frac{1}{\pi h}$ (see figure 9.5). This behaviour of c_0 is analogous to that of the Dedekind sum,

$$\mathrm{s}\Big(\frac{h}{k}\Big) = -\frac{1}{4k}\sum_{m=1}^{k}\cot\Big(\frac{\pi m}{k}\Big)\cot\Big(\frac{\pi m h}{k}\Big),$$

which satisfies the well known reciprocity formula

$$\mathrm{s}\Big(\frac{h}{k}\Big) + s\Big(\frac{k}{h}\Big) - \frac{1}{12hk} = \frac{1}{12}\Big(\frac{h}{k} + \frac{k}{h} - 3\Big). \tag{9.1.4}$$

Both these functions, as well as some others given below, are (nearly) examples of what Zagier calls a "quantum modular form". These are functions $q(x)$ of the rational numbers whose period function $q(x) - \frac{1}{x^k}q_{|\gamma}(x)$ gains some continuity properties for all $\gamma = \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in \mathrm{SL}(2,\mathbb{Z})$, where as usual $q_{|\gamma}(x) := q\big(\frac{ax+b}{cx+d}\big)$ (see [Zag]).

Corollary 9.1.2. *The numbers* $\mathrm{c}_0\big(\frac{h}{k}\big)$ *can be computed to within a prescribed accuracy in a time that is polynomial in* $\log k$.

This corollary descends immediately from the reciprocity formula (9.1.3) and, thanks to Euclid's algorithm, the same result then holds also for $V\big(\frac{h}{k}\big)$.

Theorem 9.1.1 can be generalized to the sums

$$\mathrm{c}_a\Big(\frac{h}{k}\Big) := k^a\sum_{m=1}^{k-1}\cot\Big(\frac{\pi m h}{k}\Big)\zeta\Big(-a, \frac{m}{k}\Big), \tag{9.1.5}$$

where $\zeta(s,x)$ is the Hurwitz zeta function, defined as $\zeta(s,x) := \mathcal{L}(s,0,x)$ with $\mathcal{L}$ as in (4.3.3). (Notice that when $a = -1$ the poles of $\zeta\big(-a, \frac{m}{k}\big)$ in (9.1.5) cancel).

Notice that, for all a, $\mathrm{c}_a\big(\frac{h}{k}\big)$ is odd and periodic in $x = \frac{h}{k}$ with period 1 and, for non-negative integers a, $i\,\mathrm{c}_a\big(\frac{h}{k}\big)$ takes values in the cyclotomic field of $2k$-th roots of unity.

At the non-negative integers, $a = n \geq 0$, these cotangent sums can be expressed in terms of the Bernoulli polynomials,

$$\mathrm{c}_n\Big(\frac{h}{k}\Big) = -k^n\sum_{m=1}^{k-1}\cot\Big(\frac{\pi m h}{k}\Big)\frac{B_{n+1}\big(\frac{m}{k}\big)}{n+1},$$

most interestingly in the case when n is even, since $\mathrm{c}_n \equiv 0$ for positive odd n.

If $a = -n$ is a negative integer one can write c_a as

$$\mathrm{c}_{-n}\left(\frac{h}{k}\right) = \frac{(-1)^n}{k^n(n-1)!}\sum_{m=1}^{k-1}\cot\left(\frac{\pi m h}{k}\right)\Psi\left(n-1, \frac{m}{k}\right),$$

where $\Psi(m,z) := \frac{d^{m+1}}{dz^{m+1}}\log\Gamma(z)$ is the polygamma function.

By the reflection formula for the polygamma function,

$$\Psi(m, 1-z) + (-1)^{m+1}\Psi(m,z) = (-1)^m\pi\frac{d^m}{dz^m}\cot(\pi z),$$

for a positive odd integer n we can write c_{-n} as

$$\mathrm{c}_{-n}\left(\frac{h}{k}\right) = -\frac{\pi}{2k^n(n-1)!}\sum_{m=1}^{k-1}\cot\left(\frac{\pi m h}{k}\right)\frac{d^{n-1}}{dz^{n-1}}\cot(\pi z)\bigg|_{z=\frac{m}{k}}$$

and, in particular,

$$\mathrm{c}_{-1}\left(\frac{h}{k}\right) = 2\pi s\left(\frac{h}{k}\right).$$

Like the case $a = 0$, these cotangent sums appear in the value at $s = 0$,

$$D\left(0, a, \frac{h}{k}\right) = -\frac{1}{2}\zeta(-a) + \frac{i}{2}\,\mathrm{c}_a\left(\frac{h}{k}\right), \tag{9.1.6}$$

of the function $D\big(s, a, \frac{h}{k}\big)$, defined for $\Re(s) > 1 + \max(0, \Re(a))$ by

$$D\left(s, a, \frac{h}{k}\right) := \sum_{n=1}^{\infty}\frac{\sigma_a(n)\,\mathrm{e}(nh/k)}{n^s}.$$

Moreover, the cotangent sums c_a appear also in a shifted version of Vasyunin's formula (9.1.2) (see Theorem 9.1.4 below).

Theorem 9.1.3. *Let $h, k \geq 1$, $(h,k) = 1$. Then*

$$\mathrm{c}_a\left(\frac{h}{k}\right) - \left(\frac{k}{h}\right)^{1+a}\mathrm{c}_a\left(\frac{-k}{h}\right) + k^a\frac{a\zeta(1-a)}{\pi h} = -i\zeta(-a)\psi_a\left(\frac{h}{k}\right), \tag{9.1.7}$$

where $\psi_a(z)$ is given by (7.1.4) and is analytic on $\mathbb{C}'$.

In particular, $\mathrm{c}_a\left(\frac{h}{k}\right)$ gives an example of an "imperfect" quantum modular form of weight $1+a$. We remark that these (imperfect) quantum modular forms are analogous to the "quantum Maass forms" studied by Bruggeman in [Bru], the former being associated to Eisenstein series and the latter to Maass forms. The main difference between these two classes of quantum forms comes from the fact that the L-functions associated to Maass forms are entire, while for Eisenstein series the associated L-functions are not, since they are products of two shifted Riemann zeta functions. This translates into quantum Maass forms being quantum modular forms in the strict sense, whereas the reciprocity formulas for the cotangent sums contain a non-smooth correction term.

The function g_{-n} appearing in the representation (7.1.4) of ψ_{-n} is identically zero for all negative integers n (although the reciprocity formula is trivial for negative odd integers), thus in these cases the reciprocity formulae involve only rational functions in h and k and one recovers some known reciprocity formulae for cotangent sums. In particular, for $a = -1$ the reciprocity formula (9.1.7) reduces to the reciprocity formula for the Dedekind sum (9.1.4). This has been generalized by Rademacher to a three term formula for the Dedekind sum and for c_{-n} by Beck [Beck]; in Chapter 10 we will extend this formulae to all the cotangent sums c_a.

New formulae can be obtained by differentiating (9.1.7). For example, writing

$$\mathrm{c}^*_{-1}\left(\frac{h}{k}\right) := \frac{1}{k}\sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right)\gamma_1\left(\frac{m}{k}\right),$$

where $\gamma_1(x)$ is the first generalized Stieltjes constant defined by

$$\zeta(s,x) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n(x)(s-1)^n,$$

then taking the derivative at -1 of (9.1.7) multiplied by k^{-a} we get the formula

$$\mathrm{c}^*_{-1}\left(\frac{h}{k}\right) - \mathrm{c}^*_{-1}\left(\frac{-k}{h}\right) + \frac{\zeta'(2) + \frac{\pi^2}{6}}{\pi k h} + \pi \log k \left(\frac{1}{6}\frac{k}{h} - \frac{1}{2}\right) = q\left(\frac{h}{k}\right),$$

where

$$q(z) := -\frac{1}{\pi z}\zeta'(2) + \frac{\pi}{2}(\log z + \gamma) + g'_{-1}(z)$$

is holomorphic in $\mathbb{C}'$.

In the last section of the chapter, we give a new proof of a shifted version of Vasyunin's formula.

Theorem 9.1.4. *Let* $(h,k)=1$, $h,k \geq 1$. *Let* $|\Re(a)|<1$. *Then*

$$\begin{aligned}\frac{1+a}{2\pi}\int_{-\infty}^{\infty}\zeta\Big(\frac{1}{2}+\frac{a}{2}+it\Big)\zeta\Big(\frac{1}{2}+\frac{a}{2}-it\Big)\Big(\frac{h}{k}\Big)^{-it}\frac{\mathrm{d}t}{(\frac{1}{2}+\frac{a}{2}+it)(\frac{1}{2}+\frac{a}{2}-it)} =\\ = -\frac{\zeta(1+a)}{2}\bigg(\Big(\frac{k}{h}\Big)^{\frac{1}{2}+\frac{a}{2}}+\Big(\frac{h}{k}\Big)^{\frac{1}{2}+\frac{a}{2}}\bigg)+\frac{\zeta(a)}{a}\bigg(\Big(\frac{k}{h}\Big)^{\frac{1}{2}-\frac{a}{2}}+\Big(\frac{h}{k}\Big)^{\frac{1}{2}-\frac{a}{2}}\bigg)+\\ -\Big(\frac{1}{hk}\Big)^{\frac{1}{2}+\frac{a}{2}}(2\pi)^a\Gamma(-a)\sin\frac{\pi a}{2}\bigg(\mathrm{c}_a\Big(\frac{\overline{h}}{k}\Big)+\mathrm{c}_a\Big(\frac{\overline{k}}{h}\Big)\bigg).\end{aligned} \tag{9.1.8}$$

We remark that Vasyunin's formula can be itself interpreted as a reciprocity formula for

$$\tilde{\mathrm{c}}_a\Big(\frac{h}{k}\Big) := \frac{1}{k^{1+a}}\,\mathrm{c}_a\Big(\frac{\overline{h}}{k}\Big).$$

In fact, rearranging the terms in (9.1.8) one obtains

$$\tilde{\mathrm{c}}_a\Big(\frac{h}{k}\Big)+\Big(\frac{h}{k}\Big)^{1+a}\tilde{\mathrm{c}}_a\Big(\frac{k}{h}\Big)=\mathfrak{h}_a\Big(\frac{h}{k}\Big)$$

where, for $|\Re(a)|<1$, $\mathfrak{h}(x)$ is a continuous function of $x>0$. Notice that $\tilde{\mathrm{c}}_a$ is also periodic of period 1; in particular, $\tilde{\mathrm{c}}_a$ could be thought as a quantum modular form of weight $-1-a$. However, it follows from the work of Báez-Duarte, Balazard, Landreau and Saias [BBLS05] that $\mathfrak{h}_a(x)$ is not differentiable at any rational number (for $a=0$, although their work should be easily extendible to all a with $|\Re(a)|<1$) and, in particular, the reciprocity formula can't be used to compute quickly $\tilde{\mathrm{c}}_a$ (or c_a).

Finally, we remark that one could obtain a new proof of the reciprocity formula for the Dedekind sum by letting a tend to -1 in (9.1.8) (notice that for the Dedekind sum one has $\mathrm{s}\big(\frac{h}{k}\big)=\mathrm{s}\Big(\frac{\overline{h}}{k}\Big)$).

9.2 A reciprocity formula for a cotangent sum

We start by recalling the basic properties of $D\big(s,a,\frac{h}{k}\big)$.

Lemma 9.2.1. *For (h,k)=1, $k>0$ and $a\in\mathbb{C}$,*

$$D\Big(s,a,\frac{h}{k}\Big)-k^{1+a-2s}\zeta(s-a)\zeta(s)$$

is an entire function of s. Moreover, $D\big(s,a,\frac{h}{k}\big)$ satisfies the functional equation

$$\begin{aligned}D\Big(s,a,\frac{h}{k}\Big)=&-\frac{2}{k}\Big(\frac{k}{2\pi}\Big)^{2-2s+a}\Gamma(1-s+a)\Gamma(1-s)\times\\&\times\Big(\cos\Big(\frac{\pi}{2}(2s-a)\Big)D\Big(1-s,-a,-\frac{\overline{h}}{k}\Big)+\\&-\cos\frac{\pi a}{2}\,D\Big(1-s,-a,\frac{\overline{h}}{k}\Big)\Big).\end{aligned}\tag{9.2.1}$$

Moreover

$$D\Big(0,a,\frac{h}{k}\Big)=\frac{i}{2}\,\mathrm{c}_a\Big(\frac{h}{k}\Big)-\frac{1}{2}\zeta(-a).\tag{9.2.2}$$

Proof. The analytic continuation and the functional equation for $D\big(s,a,\frac{h}{k}\big)$ can be proved easily using the analogous properties (4.3.4) for the Hurwitz zeta function and the observation that

$$D\Big(s,a,\frac{h}{k}\Big)=\frac{1}{k^{2s-a}}\sum_{m,n=1}^{k}\mathrm{e}\Big(\frac{mnh}{k}\Big)\zeta\Big(s-a,\frac{m}{k}\Big)\zeta\Big(s,\frac{n}{k}\Big).$$

Moreover, applying this equality at $s=0$, we see that

$$\begin{aligned}D\Big(0,a,\frac{h}{k}\Big)&=-k^a\sum_{m,n=1}^{k-1}\mathrm{e}\Big(\frac{mnh}{k}\Big)\zeta\Big(-a,\frac{m}{k}\Big)B_1\Big(\frac{n}{k}\Big)-\frac{\zeta(-a)}{2}\\&=\frac{i}{2}\,\mathrm{c}_a\Big(\frac{h}{k}\Big)-\frac{\zeta(-a)}{2},\end{aligned}$$

where we used

$$\sum_{n=1}^{k-1}B_1\Big(\frac{n}{k}\Big)\Big(\mathrm{e}\Big(\frac{mh}{k}\Big)\Big)^n=-\frac{1}{2}\frac{1+\mathrm{e}\big(\frac{mh}{k}\big)}{1-\mathrm{e}\big(\frac{mh}{k}\big)}=-\frac{i}{2}\cot\Big(\frac{\pi mh}{k}\Big),$$

that can be easily obtained from the equality

$$B_1(x)=\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{te^{xt}}{e^t-1}\Big)\Big|_{t=0}.$$

□

We can now prove Theorem 9.1.3, Theorem 9.1.1 will then follow after taking the limit $a \to 0$.

Proof of Theorem 9.1.3. Firstly, observe that we can assume $0 \neq |a| < 1$, since the result extends to all a by analytic continuation. Now, taking $z = \frac{h}{k}(1+i\delta)$, with $\delta > 0$, we have

$$\begin{aligned}\mathcal{S}_a(z) &= \sum_{n\geq 1} \sigma_a(n)\, \mathrm{e}\left(n\frac{h}{k}\right) e^{-2\pi n \frac{h}{k}\delta} \\ &= \frac{1}{2\pi i}\int_{(2)} \Gamma(s) D\left(s, a, \frac{h}{k}\right)\left(2\pi\frac{h}{k}\delta\right)^{-s} \mathrm{d}s.\end{aligned}$$

Therefore, moving the line of integration to $\sigma = -\frac{1}{2}$, by the residue theorem we have

$$\mathcal{S}_a(z) = \frac{k^a}{2\pi h\delta}\zeta(1-a) + \frac{1}{(2\pi h\delta)^{1+a}}\zeta(1+a)\Gamma(1+a) + D\left(0, a, \frac{h}{k}\right) + O\left(\delta^{\frac{1}{2}}\right).$$

Similarly,

$$\begin{aligned}\frac{1}{z^{1+a}}\mathcal{S}_a\left(\frac{-1}{z}\right) &= \frac{1}{z^{1+a}}\sum_{n\geq 1}\sigma_a(n)\,\mathrm{e}\left(-n\frac{k}{h}\right)e^{-2\pi\frac{k}{h}\frac{\delta}{1+i\delta}} \\ &= \frac{k^a}{2\pi\delta h}\zeta(1-a) + \frac{1}{(2\pi\delta h)^{1+a}}\zeta(1+a)\Gamma(1+a) \\ &\quad - ia\frac{k^a}{2\pi h}\zeta(1-a) + \left(\frac{k}{h(1+i\delta)}\right)^{1+a} D\left(0, a, -\frac{k}{h}\right) + O\left(\delta^{\frac{1}{2}}\right).\end{aligned}$$

In particular, as δ goes to 0^+, we have

$$\begin{aligned}\mathcal{S}_a(z) - \frac{1}{z^{1+a}}\mathcal{S}_a\left(\frac{-1}{z}\right) \longrightarrow & D\left(0, a, \frac{h}{k}\right) - \left(\frac{k}{h}\right)^{1+a} D\left(0, a, -\frac{k}{h}\right) + \\ & + ia\frac{k^a}{2\pi h}\zeta(1-a).\end{aligned}$$

Applying Theorem 7.1.1, it follows that

$$\begin{aligned}D\left(0, a, \frac{h}{k}\right) - \left(\frac{k}{h}\right)^{1+a} D\left(0, a, -\frac{k}{h}\right) + ia\frac{k^a}{2\pi h}\zeta(1-a) = \\ = \frac{\zeta(-a)}{2}\left(\left(\frac{k}{h}\right)^{1+a} - 1 + \psi_a\left(\frac{h}{k}\right)\right),\end{aligned}$$

which is equivalent to (9.1.7). $\square$

9.3 Analytic proof of Vasyunin's formula

Proof of Theorem 9.1.4. We need to evaluate

$$\frac{1+a}{2\pi(hk)^{\frac{1}{2}+\frac{a}{2}}}\int_{-\infty}^{\infty}\zeta\Big(\frac{1}{2}+\frac{a}{2}+it\Big)\zeta\Big(\frac{1}{2}+\frac{a}{2}-it\Big)\times$$

$$\times\Big(\frac{h}{k}\Big)^{it}\frac{\mathrm{d}t}{(\frac{1}{2}+\frac{a}{2}+it)(\frac{1}{2}+\frac{a}{2}-it)}=$$

$$=\frac{1+a}{2\pi i}\int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)}\frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}}\frac{\mathrm{d}s}{(s+a)(1-s)}.$$

We rewrite this as

$$\frac{1+a}{2\pi i}\int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)}\frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}}\frac{\mathrm{d}s}{(s+a)(1-s)}=$$

$$=\frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)}\frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}}\frac{\mathrm{d}s}{1-s}+$$

$$+\frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)}\frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}}\frac{\mathrm{d}s}{s+a}$$

$$=I_a\Big(\frac{h}{k}\Big)+I_a\Big(\frac{k}{h}\Big),$$

where

$$I_a\Big(\frac{h}{k}\Big):=\frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)}\frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}}\frac{\mathrm{d}s}{1-s}.$$

The integral is not absolutely convergent, so some care is needed. One could introduce a convergence factor $e^{\delta s^2}$ and let $\delta\to 0^+$ at the end of the argument, or one could work with the understanding that the integrals are to be interpreted as $\lim_{T\to\infty}\int_{c-iT}^{c+iT}$. We opt for the latter. Recall that $\zeta(s)=\chi(s)\zeta(1-s)$, where

$$\chi(1-s)=\big((2\pi i s)^{-s}+(-2\pi i s)^{-s}\big)\Gamma(s).$$

This leads to

$$\frac{1}{2\pi i}\int_{(2)}\frac{\chi(1-s)}{1-s}u^{-s}\,\mathrm{d}s=\frac{-1}{2\pi i}\int_{(2)}\big((2\pi i s)^{-s}+(-2\pi i s)^{-s}\big)\frac{\Gamma(s)}{s-1}u^{-s}\,\mathrm{d}s$$

$$=\frac{-1}{2\pi i u}\int_{(1)}\big((2\pi i s)^{-s-1}+(-2\pi i s)^{-s-1}\big)\Gamma(s)u^{-s}\,\mathrm{d}s$$

$$=\frac{\sin 2\pi u}{\pi u}.$$

Using Cauchy's theorem, the functional equation for $\zeta(s)$, and the Dirichlet series for $\zeta(s+a)\zeta(s)$, we have

$$\begin{aligned}I_a\Big(\frac{h}{k}\Big) &= -\operatorname{Res}_{s=1}\frac{\chi(1-s)\zeta(s+a)\zeta(s)}{h^{s+a}k^{1-s}(1-s)} - \operatorname{Res}_{s=1-a}\frac{\chi(1-s)\zeta(s+a)\zeta(s)}{h^{s+a}k^{1-s}(1-s)}+\\ &\quad+\frac{1}{\pi h^{1+a}}\sum_{n=1}^{\infty}\frac{\sigma_{-a}(n)\sin 2\pi n\frac{h}{k}}{n}\\ &= -\frac{\zeta(1+a)}{2h^{1+a}}+\frac{\zeta(a)}{ahk^a}+\frac{1}{\pi h^{1+a}}\sum_{n=1}^{\infty}\sin\Big(2\pi n\frac{h}{k}\Big)\frac{\sigma_{-a}(n)}{n}.\end{aligned}$$

By the functional equation (9.2.1) for D we see that

$$\begin{aligned}\frac{D\big(s,-a,\frac{h}{k}\big)-D\big(s,-a,-\frac{h}{k}\big)}{2i} &= \frac{2}{k}\Big(\frac{k}{2\pi}\Big)^{2-2s-a}\Gamma(1-s-a)\Gamma(1-s)\times\\ \times\Big(\cos\Big(\frac{\pi}{2}(2s+a)\Big)+\cos\frac{\pi a}{2}\Big)&\Big(D\Big(1-s,a,\frac{\overline{h}}{k}\Big)-D\Big(1-s,a,-\frac{\overline{h}}{k}\Big)\Big),\end{aligned}$$

so that, defining

$$D_{\sin}\Big(s,-a,\frac{h}{k}\Big):=\sum_{n=1}^{\infty}\sin\Big(2\pi n\frac{h}{k}\Big)\frac{\sigma_{-a}(n)}{n^s},$$

we have

$$\begin{aligned}D_{\sin}\Big(s,-a,\frac{h}{k}\Big) &= \frac{2}{k}\Big(\frac{k}{2\pi}\Big)^{2-2s-a}\Gamma(1-s-a)\Gamma(1-s)\times\\ &\quad\times\Big(\cos\Big(\frac{\pi}{2}(2s+a)\Big)+\cos\frac{\pi a}{2}\Big)D_{\sin}\Big(1-s,a,\frac{\overline{h}}{k}\Big).\end{aligned}\tag{9.3.1}$$

In particular, $D_{\sin}\big(s,-a,\frac{h}{k}\big)$ is regular at $s=1$. Noting that

$$\lim_{s\to1}\Gamma(1-s-a)\Gamma(1-s)\Big(\cos\Big(\frac{\pi}{2}(2s+a)\Big)+\cos\frac{\pi a}{2}\Big)=-\pi\Gamma(-a)\sin\frac{\pi a}{2}$$

and

$$D_{\sin}\Big(0,a,\frac{\overline{h}}{k}\Big)=\frac{1}{2}\,\mathrm{c}_a\Big(\frac{\overline{h}}{k}\Big),$$

we obtain, by letting $s\to1$ in (9.3.1), the identity

$$D_{\sin}\Big(1,-a,\frac{h}{k}\Big)=2^a\Big(\frac{\pi}{k}\Big)^{1+a}\Gamma(-a)\sin\frac{\pi a}{2}\,\mathrm{c}_a\Big(\frac{\overline{h}}{k}\Big),$$

whence

$$\sum_{n=1}^{\infty} \frac{\sigma_{-a}(n) \sin 2\pi n \frac{h}{k}}{\pi n h^{1+a}} = -\left(\frac{1}{hk}\right)^{1+a} (2\pi)^a \Gamma(-a) \sin \frac{\pi a}{2} \, \mathrm{c}_a\left(\frac{\overline{h}}{k}\right).$$

Thus,

$$I_a\left(\frac{h}{k}\right) = -\frac{\zeta(1+a)}{2h^{1+a}} + \frac{\zeta(a)}{ahk^a} - \left(\frac{1}{hk}\right)^{1+a} (2\pi)^a \Gamma(-a) \sin \frac{\pi a}{2} \, \mathrm{c}_a\left(\frac{\overline{h}}{k}\right)$$

and the theorem follows. □

Chapter 10

A generalization of Rademacher's reciprocity law

The work presented in this chapter was first published in [Bet1].

10.1 Introduction

In Chapter 9, we introduced the cotangent sums

$$\mathrm{c}_a\left(\frac{h}{k}\right) := k^a \sum_{m=1}^{k-1} \cot\left(\frac{\pi h m}{k}\right) \zeta\left(-a, \frac{m}{k}\right),$$

for a rational number $\frac{h}{k}$, with $(h,k) = 1$, $k > 1$, and a complex number a (where c_0 has to be interpreted as $\mathrm{c}_0\left(\frac{h}{k}\right) := \lim_{a\to 0} \mathrm{c}_a\left(\frac{h}{k}\right)$).

The most interesting cases arise when $a = -1$ and $a = 0$. In the former case, c_{-1} is, up to a constant, the Dedekind sum,

$$s\left(\frac{h}{k}\right) := \frac{1}{4k} \sum_{m=1}^{k-1} \cot\left(\frac{\pi h m}{k}\right) \cot\left(\frac{\pi m}{k}\right) = \frac{1}{2\pi}\, \mathrm{c}_{-1}\left(\frac{h}{k}\right).$$

The main property of the Dedekind sum is that it satisfies a reciprocity formula

$$s\left(\frac{h}{k}\right) + s\left(\frac{k}{h}\right) - \frac{1}{12hk} = \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} - 3\right), \tag{10.1.1}$$

for $(h,k)=1$, $h,k\in\mathbb{N}^+$. This formula, due to Dedekind, has been generalized by Rademacher, who proved that

$$s\biggl(\frac{a\bar{b}}{c}\biggr)+s\biggl(\frac{b\bar{c}}{a}\biggr)+s\biggl(\frac{c\bar{a}}{b}\biggr)=\frac{a^2+b^2+c^2}{12abc}-\frac{1}{4}, \tag{10.1.2}$$

for $(a,b)=(b,c)=(a,c)=1$, $a,b,c\in\mathbb{N}^+$, and where $\overline{b}$ (respectively $\overline{c},\overline{a}$) denotes the inverse of b (resp. c,a) modulo c (resp. a,b).

For $a=0$, one has the cotangent sum

$$\mathrm{c}_0\biggl(\frac{h}{k}\biggr)=\sum_{m=1}^{k-1}\Bigl\{\frac{m}{k}\Bigr\}\cot\biggl(\frac{\pi mh}{k}\biggr),$$

which is relevant to the Nyman-Beurling-Báez-Duarte approach to the Riemann hypothesis. In Chapter 9 we showed that (10.1.1) can be generalized to this function. In this chapter we provide the analogue of (a generalization of) (10.1.2) to c_0 and to the whole family of cotangent sums c_a.

Before stating the theorem we recall the definition of the Estermann function $D\bigl(s,a,\frac{h}{k}\bigr)$, which is initially defined for $\Re(s)>1+\max(0,\Re(a))$ as

$$D\biggl(s,a,\frac{h}{k}\biggr):=\sum_{n=1}^{\infty}\frac{\sigma_a(n)\,\mathrm{e}(nh/k)}{n^s}$$

and can be analytically continued to $\mathbb{C}\setminus\{1,1+a\}$. Moreover, $D\bigl(s,a,\frac{h}{k}\bigr)$ grows at most as a polynomial in vertical strips.

Theorem 10.1.1. *Let $a\in\mathbb{C}$ and let M be any integer greater than or equal to $-\frac{1}{2}\min(0,\Re(a))$. Let $h,k,p,q\in\mathbb{N}^+$, with $(h,k)=(p,q)=1$, and let $d=(pk+h,q)$. Then*

$$\begin{aligned}\mathrm{c}_a\biggl(\frac{pk+h}{qk}\biggr)-\biggl(\frac{k}{h}\biggr)^{1+a}\mathrm{c}_a\biggl(\frac{-\overline{p}h-k}{qh}\biggr)-\mathrm{c}_a\biggl(\frac{p}{q}\biggr)+a\zeta(1-a)\frac{(kq)^a d^{1-a}}{\pi h}&=\\ =-2i\sum_{m=1}^{2M}D\biggl(-m,a,\frac{p}{q}\biggr)\frac{\Bigl(2\pi i\frac{h}{kq}\Bigr)^m}{m!}+g_{a,M}\biggl(\frac{h}{k},\frac{p}{q}\biggr)+&\\ +2\biggl(2\pi\frac{h}{k}\biggr)^{-1}q^a\zeta(1-a)-\cot\frac{\pi a}{2}\zeta(-a)\biggl(\frac{k}{h}\biggr)^{1+a},&\end{aligned} \tag{10.1.3}$$

where $\overline{p}$ is the inverse of p modulo q and

$$g_{a,M}\left(z,\frac{h}{k}\right):=\frac{1}{\pi i}\int_{\left(-\frac{1}{2}-2M\right)}\Gamma(s)\frac{\cos\frac{\pi a}{2}}{\sin\pi(s-a)}\times$$
$$\times\left(e^{-\frac{\pi i}{2}(s-a)}D\left(s,a,\frac{h}{k}\right)+D\left(s,a,-\frac{h}{k}\right)e^{\frac{\pi i(s-a)}{2}}\right)\left(\frac{2\pi z}{k}\right)^{-s}\mathrm{d}s. \tag{10.1.4}$$

In particular, for all $(p,q)=1$, the left hand side of (10.1.3) *can be continued to a function of $\frac{h}{k}$ which is holomorphic on $\mathbb{C}':=\mathbb{C}\setminus\mathbb{R}_{\le 0}$.*

Corollary 10.1.2. *Let $h,k,p,q\in\mathbb{N}^+$, with $(h,k)=(p,q)=1$, and let $d=(pk+h,q)$. Let $\overline{p}$ be the inverse of p modulo q. Then*

$$\mathrm{c}_0\left(\frac{pk+h}{qk}\right)+\frac{k}{h}\,\mathrm{c}_0\left(\frac{\overline{p}h+k}{qh}\right)-\mathrm{c}_0\left(\frac{p}{q}\right)-\frac{d}{\pi h}=f\left(\frac{h}{k},\frac{p}{q}\right),$$

where

$$f\left(z,\frac{p}{q}\right):=-\frac{\log(2\pi qz)-\gamma}{\pi z}+$$
$$+\frac{1}{\pi i}\int_{\left(-\frac{1}{2}\right)}\frac{\Gamma(s)}{\sin\pi s}\left(e^{-\frac{\pi is}{2}}D\left(s,0,\frac{p}{q}\right)+e^{\frac{\pi is}{2}}D\left(s,0,-\frac{p}{q}\right)\right)\left(2\pi\frac{z}{q}\right)^{-s}\mathrm{d}s$$

is a holomorphic function of z on $\mathbb{C}'$.

In the case of $a=-1$ Theorem 10.1.1 yields the following corollary.

Corollary 10.1.3. *Let $h,k,p,q\in\mathbb{N}^+$, with $(h,k)=(p,q)=1$, and $d=(pk+h,q)$. Then*

$$s\left(\frac{pk+h}{qk}\right)+s\left(\frac{\overline{p}h+k}{qh}\right)-s\left(\frac{p}{q}\right)=\frac{(k^2+d^2+h^2)}{12hkq}-\frac{1}{4}. \tag{10.1.5}$$

This is an extension of Rademacher's formula and is equivalent to Lemma 7 of [CFKS] (which is based on (26) of [HH]), as we shall show at the end of Section 10.3. Finally, it should be noticed that for negative odd integer a, the identities we obtain involve, like in the case when $a=-1$, only cotangent sums and a rational function and are particular cases of the formulae obtained by Beck in [Beck].

10.2 The twisted period function

The following lemma extends (7.1.7) to the case of

$$\mathcal{S}\Big(z,a,\frac{h}{k}\Big) := \sum_{n=1}^{\infty} \sigma_a(n)\,\mathrm{e}\Big(n\frac{h}{k}\Big)\,\mathrm{e}(nz), \tag{10.2.1}$$

where $\Re(z) > 0$.

Lemma 10.2.1. *Let $a \in \mathbb{C}$ and let M be any fixed integer greater than or equal to $-\frac{1}{2}\min(0,\Re(a))$. Then*

$$\mathcal{S}\Big(\frac{z}{k},a,\frac{h}{k}\Big) - \frac{1}{z^{1+a}}\mathcal{S}\Big(-\frac{1}{kz},a,\frac{-\overline{h}}{k}\Big)$$

extends to an analytic function of z on $\mathbb{C}' := \mathbb{C}\setminus\mathbb{R}_{\leq 0}$ via the representation

$$\mathcal{S}\Big(\frac{z}{k},a,\frac{h}{k}\Big) - \frac{1}{z^{1+a}}\mathcal{S}\Big(-\frac{1}{kz},a,\frac{-\overline{h}}{k}\Big) = r_{a,M}\Big(z,\frac{h}{k}\Big) + \frac{i}{2}g_{a,M}\Big(z,\frac{h}{k}\Big),$$

where $g_{a,M}\big(z,\frac{h}{k}\big)$ is as in (10.1.4) *and*

$$\begin{aligned} r_{a,M}\Big(z,\frac{h}{k}\Big) := {}& ik^a\frac{\zeta(1-a)}{2\pi z} + e^{\frac{\pi i(1+a)}{2}}\Gamma(1+a)\frac{\zeta(1+a)}{(2\pi z)^{1+a}} \\ & + \sum_{m=1}^{2M} D\Big(-m,a,\frac{h}{k}\Big)\frac{i^m}{m!}(2\pi z/k)^m + D\Big(0,a,\frac{h}{k}\Big). \end{aligned}$$

Proof. Firstly observe that we can assume $0 \neq |\Re(a)| < 1$, since the lemma will then follow by analytic continuation in a. Now, we have that $\mathcal{S}\big(z,a,\frac{h}{k}\big)$ can be written as

$$\mathcal{S}\Big(\frac{z}{k},a,\frac{h}{k}\Big) = \frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))} D\Big(s,a,\frac{h}{k}\Big)e^{\frac{\pi i s}{2}}\Gamma(s)(2\pi z/k)^{-s}\,\mathrm{d}s$$

and, by contour integration, this is equal to

$$\begin{aligned} \mathcal{S}\Big(\frac{z}{k},a,\frac{h}{k}\Big) = {}& \frac{1}{2\pi i}\int_{\left(-\frac{1}{2}-2M\right)} D\Big(s,a,\frac{h}{k}\Big)e^{\frac{\pi i s}{2}}\Gamma(s)(2\pi z/k)^{-s}\,\mathrm{d}s+ \\ & + r_{a,M}\Big(\frac{z}{k},\frac{h}{k}\Big). \end{aligned} \tag{10.2.2}$$

Now, consider

$$
\begin{aligned}
\frac{1}{(zk)^{1+a}}\,\mathcal{S}\Big(-\frac{1}{zk},a,-\frac{\overline{h}}{k}\Big) &= \\
&= \frac{1}{(zk)^{1+a}}\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))} D\Big(s,a,-\frac{\overline{h}}{k}\Big)\Gamma(s)e^{\frac{\pi i s}{2}}\Big(2\pi\frac{-1}{zk}\Big)^{-s}\,\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))} D\Big(s,a,-\frac{\overline{h}}{k}\Big)\Gamma(s)e^{-\frac{\pi i s}{2}}(2\pi)^{-s}(zk)^{s-1-a}\,\mathrm{d}s,
\end{aligned}
$$

since in this context $0 < \arg z < \pi$ and $0 < \arg\frac{-1}{z} < \pi$, so the identity $\arg\frac{-1}{z} = \pi - \arg z$ holds. Applying the functional equation (9.2.1), we get that this is

$$
\begin{aligned}
&-\frac{2}{k}\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}\Big(\frac{k}{2\pi}\Big)^{2-2s+a}\Gamma(1-s+a)\Gamma(1-s)\times \\
&\quad\times\Big(\cos\Big(\frac{\pi}{2}(2s-a)\Big)D\Big(1-s,-a,\frac{h}{k}\Big)-\cos\frac{\pi a}{2}\,D\Big(1-s,-a,-\frac{h}{k}\Big)\Big)\times \\
&\quad\times\Gamma(s)\,e^{-\frac{\pi i s}{2}}(2\pi)^{-s}(zk)^{s-1-a}\,\mathrm{d}s.
\end{aligned}
$$

Now, observing that $D\big(s,-a,-\frac{h}{k}\big) = D\big(s+a,a,-\frac{h}{k}\big)$ and using Euler's reflection formula, we get that this is equal to

$$
\begin{aligned}
&-\frac{2\pi}{k}\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}\Big(\frac{k}{2\pi}\Big)^{2-2s+a}\Gamma(1-s+a)\times \\
&\qquad\times\Big(\cos\Big(\frac{\pi}{2}(2s-a)\Big)D\Big(1-s+a,a,\frac{h}{k}\Big)+ \\
&\qquad\qquad-\cos\frac{\pi a}{2}\,D\Big(1-s+a,a,-\frac{h}{k}\Big)\Big)\frac{e^{-\frac{\pi i s}{2}}}{\sin\pi s}(2\pi)^{-s}(zk)^{s-1-a}\,\mathrm{d}s.
\end{aligned}
$$

Now, we make the change of variable $s \to 1-s+a$ and then move the line of integration to $-\frac{1}{2}-2M$ without crossing any pole. Thus, we get

$$
\begin{aligned}
\frac{1}{(zk)^{1+a}}\mathcal{S}\Big(-\frac{1}{zk},a,\frac{\overline{h}}{k}\Big) &= -\frac{i}{k}\frac{1}{2\pi i}\int_{\left(-\frac{1}{2}-2M\right)} k^{-a}\Gamma(s)\frac{e^{\frac{\pi i(s-a)}{2}}}{\sin\pi(s-a)}(2\pi z/k)^{-s}\times \\
&\quad\times\Big(\cos\Big(\frac{\pi}{2}(2s-a)\Big)D\Big(s,a,\frac{h}{k}\Big)+\cos\frac{\pi a}{2}\,D\Big(s,a,-\frac{h}{k}\Big)\Big)\,\mathrm{d}s.
\end{aligned}
\tag{10.2.3}
$$

The lemma then follows by taking the difference between (10.2.2) and (10.2.3), thanks to the identity

$$
e^{\frac{\pi i s}{2}} + i\frac{\cos\big(\frac{\pi}{2}(2s-a)\big)}{\sin\pi(s-a)}\,e^{\frac{\pi i(s-a)}{2}} = i\frac{e^{-\frac{\pi i}{2}(s-a)}\cos\frac{\pi a}{2}}{\sin\pi(s-a)}. \qquad \square
$$

10.3 A generalization of Rademacher's formula

We can now give an extension of Rademacher's reciprocity formula to the sum $\mathrm{c}_a\left(\frac{h}{k}\right)$. The proof follows the method used to prove Theorem 9.1.3.

Proof of Theorem 10.1.1. Firstly observe that we can assume $0 \neq |\Re(a)| < 1$, since the result will then follow by analytic continuation in a.

Let $z = \frac{h}{k}(1+i\xi)$ for a small $\xi > 0$ and let $\alpha = pk + h$, $\beta = qk$. We have

$$\begin{aligned}\mathcal{S}\left(\frac{z}{q}, a, \frac{p}{q}\right) &= \sum_{n=1}^{\infty} \sigma_a(n) \mathrm{e}\left(n\frac{\alpha}{\beta}\right) \mathrm{e}\left(in\frac{h}{\beta}\xi\right)\\ &= \frac{1}{2\pi i}\int_{(2+\max(\Re(a)),0)} \Gamma(s) D\left(s, a, \frac{\alpha}{\beta}\right)\left(2\pi\frac{h}{\beta}\xi\right)^{-s} \mathrm{d}s.\end{aligned}$$

Moving the line of integration to $\Re(s) = -\frac{1}{2}$ and picking up the residue encountered, by Lemma 9.2.1 we get that this is equal to

$$\begin{aligned}\mathcal{S}\left(\frac{z}{q}, a, \frac{p}{q}\right) = (\beta/d)^{a-1}\zeta(1-a)\left(2\pi\frac{h}{\beta}\xi\right)^{-1} + \frac{i}{2}\mathrm{c}_a\left(\frac{\alpha}{\beta}\right) - \frac{1}{2}\zeta(-a)+ \\ + \Gamma(1+a)(\beta/d)^{-1-a}\zeta(1+a)\left(2\pi\frac{h}{\beta}\xi\right)^{-1-a} + O(\xi^{\frac{1}{2}}).\end{aligned} \tag{10.3.1}$$

In the same way, writing

$$-\frac{1}{z} = -\frac{k}{h}(1-i\xi'), \qquad \xi' = \frac{\xi}{1+i\xi} = \xi - i\xi^2 + O(\xi^3)$$

and $\alpha' = -\overline{p}h - k$, $\beta' = qh$ (note that $(p,q) = (h,k) = (\alpha, q) = 1$ implies $(\alpha', q) = 1$), we have

$$\begin{aligned}\mathcal{S}\left(-\frac{1}{qz}, a, -\frac{\overline{p}}{q}\right) &= \sum_{n=1}^{\infty} \sigma_a(n) \mathrm{e}\left(n\frac{\alpha'}{\beta'}\right) \mathrm{e}\left(in\frac{k}{\beta'}\xi'\right)\\ &= (\beta'/d)^{a-1}\zeta(1-a)\left(2\pi\frac{k}{\beta'}\xi'\right)^{-1} + \frac{i}{2}\mathrm{c}_a\left(\frac{\alpha'}{\beta'}\right) - \frac{1}{2}\zeta(-a)\\ &\quad + \Gamma(1+a)(\beta'/d)^{-1-a}\zeta(1+a)\left(2\pi\frac{k}{\beta'}\xi'\right)^{-1-a} + O(\xi'^{\frac{1}{2}})\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{S}\left(-\frac{1}{qz}, a, -\frac{\overline{p}}{q}\right) = (\beta'/d)^{a-1}\zeta(1-a)\left(2\pi\frac{k}{\beta'}\xi\right)^{-1}(1+i\xi) + \frac{i}{2}\mathrm{c}_a\left(\frac{\alpha'}{\beta'}\right) - \frac{1}{2}\zeta(-a)\\ + \Gamma(1+a)(\beta'/d)^{-1-a}\zeta(1+a)\left(2\pi\frac{k}{\beta'}\xi\right)^{-1-a}(1+i\xi)^{1+a} + O(\xi^{\frac{1}{2}}).\end{aligned} \tag{10.3.2}$$

Therefore, from (10.3.1) and (10.3.2) it follows that

$$\begin{aligned}\mathcal{S}\left(\frac{z}{q},a,\frac{p}{q}\right)-\frac{1}{z^{1+a}}\mathcal{S}\left(-\frac{1}{qz},a,-\frac{\overline{p}}{q}\right)&=\frac{i}{2}\,\mathrm{c}_a\left(\frac{\alpha}{\beta}\right)-\frac{1}{z^{1+a}}\frac{i}{2}\,\mathrm{c}_a\left(\frac{\alpha'}{\beta'}\right)+\\ &-\frac{1}{2}\zeta(-a)+ia\zeta(1-a)\frac{(kq)^a d^{1-a}}{2\pi h}+\frac{1}{z^{1+a}}\frac{1}{2}\zeta(-a)+O(\xi^{\frac{1}{2}})\end{aligned}$$

and thus

$$\begin{aligned}\lim_{\xi\to 0^+}\mathcal{S}\left(\frac{z}{q},a,\frac{p}{q}\right)-\frac{1}{z^{1+a}}\mathcal{S}\left(-\frac{1}{qz},a,-\frac{\overline{p}}{q}\right)&=\frac{i}{2}\,\mathrm{c}_a\left(\frac{\alpha}{\beta}\right)-\left(\frac{k}{h}\right)^{1+a}\frac{i}{2}\,\mathrm{c}_a\left(\frac{\alpha'}{\beta'}\right)+\\ &-\frac{1}{2}\zeta(-a)+ia\zeta(1-a)\frac{(kq)^a d^{1-a}}{2\pi h}+\left(\frac{k}{h}\right)^{1+a}\frac{1}{2}\zeta(-a).\end{aligned}$$

By Lemma 10.2.1, this is also equal to $r_{a,M}\left(\frac{h}{k},\frac{p}{q}\right)+\frac{i}{2}g_{a,M}\left(\frac{h}{k},\frac{p}{q}\right)$ and the theorem then follows after using the functional equation for the Riemann zeta-function. $\square$

Corollary 10.1.2 follows immediately by applying Theorem 10.1.1 to the case $a=0$. We remark that replacing k with qk in Corollary 10.1.2, we obtain, for all $M\in\mathbb{Z}_{\geq 0}$,

$$\begin{aligned}c_0\left(\frac{pqk+h}{q^2k}\right)+\frac{qk}{h}c_0\left(\frac{ph+kq}{qh}\right)-c_0\left(\frac{\overline{p}}{q}\right)-\frac{1}{\pi h}&=\\ =\frac{q}{\pi^2}\sum_{m=1}^{M}(-1)^m(2m)!D_{\sin}\left(1+2m,\frac{p}{q}\right)\left(\frac{h}{2\pi k}\right)^{2m}&+q\mu_M\left(\frac{h}{k},\frac{p}{q}\right),\end{aligned}\tag{10.3.3}$$

where $\mu_M(x,y)$ is holomorphic in x for $x\in\mathbb{C}'$ and $C^{2M+1}(\mathbb{R})$ in y, and where

$$D_{\sin}(s,x):=\frac{D(s,0,x)-D(s,0,-x)}{2i},$$

for $\Re(s)>1+\max(0,\Re(a))$.

Applying Theorem 10.1.1 to $a=-1$, one obtains immediately the generalization 10.1.5 of Rademacher's reciprocity formula, since for $a=-1$ one has that $g_{-a,M}$ is identically zero.

We conclude the chapter by showing how to obtain Lemma 7 of [CFKS] from (10.1.5). This lemma states that, if $a,c,\ell,m\in\mathbb{N}^+$, with $(a,c)=(\ell,m)=$

1, and b, d are such that $ad - bc = 1$, then

$$s\left(\frac{a}{c}\right) + s\left(\frac{\ell}{m}\right) - s\left(\frac{x}{y}\right) = \frac{c^2+m^2+y^2}{12cmy} - \frac{1}{4}, \tag{10.3.4}$$

where $x = a\ell + bm$ and $y = c\ell + dm$.

To prove this result we apply Corollary 10.1.3 to $p = x$, $q = y$, $k = c/u$ and $h = m/u$, where $u = (c, m)$. We have that

$$u(pk+h) = xc + m = ac\ell + m(bc+1) = ac\ell + ad = ay = aq \tag{10.3.5}$$

and

$$\begin{aligned}\ell(\overline{p}h+k)u &= \ell(\overline{p}m+c) = \ell\overline{p}m + \ell c = \ell\overline{p}m + q - dm = (\ell\overline{p} - d)m + q = \\ &= ((\overline{p}p-1)d - \overline{p}bq)m + q,\end{aligned}$$

where we used $\ell = dp - bq$ which can be obtained from the definition of x and y and the condition $ad - bc = 1$. Therefore

$$\ell(\overline{p}h+k)u/q \equiv 1 \pmod{m}. \tag{10.3.6}$$

Thus (10.3.4) follows from (10.3.5) and (10.3.6) by observing that

$$s\left(\frac{\overline{\ell}}{m}\right) = s\left(\frac{\ell}{m}\right).$$

Chapter 11

An exact formula for the second moment of $\zeta(s)$

The work presented in this chapter is joint with J.B. Conrey and was first published in [BC2].

11.1 Introduction

In Part A, Chapter 3, we described the importance of understanding the asymptotics behaviour of the moments of $\zeta(s)$,

$$I_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \mathrm{d}t$$

and we introduced the moments conjecture

$$I_k(T) \sim C_k T \log^{k^2} T, \tag{11.1.1}$$

as $T \to \infty$, for some constants C_k. Instead of dealing directly with the $2k$-th moment one can search for the asymptotics of the Laplace transform

$$L_k(\delta) := \int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} e^{-\delta t}\, \mathrm{d}t,$$

for δ that goes to 0 (with $\Re(\delta) > 0$). In fact, in Section 7 of his book on the Riemann Zeta function [Tit], Titchmarsh proves that (11.1.1) is equivalent to

$$L_k(\delta) \sim C_k \frac{1}{\delta} \log^{k^2} \frac{1}{\delta}.$$

and therefore, for what concerns asymptotics, it is equivalent to study $I_k(T)$ or $L_k(\delta)$. Thanks to the smoothing factor $e^{-\delta t}$, the asymptotic behavior of $L_k(\delta)$ is better understood than that of $I_k(T)$. In fact, for the former, Kober proved that one has the asymptotic formula

$$L_1(\delta) = \frac{\gamma - \log 2\pi\delta}{2\sin\frac{\delta}{2}} + \sum_{n=0}^{N} c_n \delta^n + O\big(\delta^{N+1}\big),$$

with the implicit constant depending on N and the c_n being constants.

Atkinson [Atk] investigated the case when δ is complex and, as noted by Jutila [Jut98], his argument implies the exact formula

$$L_1(\delta) = -ie^{\frac{i\delta}{2}}\Big(\log 2\pi - \gamma + i\Big(\frac{\pi}{2} - \delta\Big)\Big) + 2\pi e^{\frac{i\delta}{2}} \sum_{n=1}^{\infty} d(n)\,\mathrm{e}\big(-ne^{-i\delta}\big) + \lambda(\delta), \tag{11.1.2}$$

where $\lambda(\delta)$ is analytic in the strip $|\Re(\delta)| < \pi$. (Notice, however, that the main term is not exhibited by this formula).

Finally, in his book on spectral theory Motohashi studied the more general integral

$$\mathfrak{L}_k(g) := \int_{-\infty}^{\infty} \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^{2k} g(t)\,\mathrm{d}t,$$

where $g(t)$ satisfies some decay conditions, obtaining, for the cases $k = 1, 2$, an exact formula in terms of conditionally convergent series (see Theorems 4.1 and 4.2 of [Mot]). See also Ivič's paper [Ivi] for other results on the topic.

Here we give another exact formula for $L_1(\delta)$, which is both asymptotic and absolutely convergent.

Theorem 11.1.1. *For $0 < \Re(\delta) < \pi$, we have*

$$L_1(\delta) = \frac{\gamma - \log 2\pi\delta}{2\sin\frac{\delta}{2}} + \frac{\pi i}{\sin\frac{\delta}{2}} \mathcal{S}_0\Big(\frac{-1}{1 - e^{-i\delta}}\Big) + h(\delta) + k(\delta),$$

where $\mathcal{S}_0(z)$ *is defined in* (7.1.1)*,* $k(\delta)$ *is analytic in* $|\Re(\delta)| < \pi$ *and* $h(\delta)$ *is* C^∞ *in* $\mathbb{R}$ *and holomorphic in*

$$\mathbb{C}'' := \mathbb{C} \setminus \{x + iy \in \mathbb{C} \mid x \in 2\pi\mathbb{Z},\ y \geq 0\}.$$

Moreover, $h(0) = 0$ *and, if* $\Im(\delta) \leq 0$*,*

$$h(\delta) = i \sum_{n \geq 0} h_n e^{-i\left(n+\frac{1}{2}\right)\delta},$$

with

$$h_n = 2^{\frac{7}{4}} \pi^{\frac{1}{4}} \frac{e^{-2\sqrt{\pi n}}}{n^{\frac{1}{4}}} \sin\left(2\sqrt{\pi n} + \frac{5\pi}{8}\right) + O\left(\frac{e^{-2\sqrt{\pi n}}}{n^{\frac{3}{4}}}\right),$$

as $n \to \infty$*.*

The most remarkable aspect of this theorem lies in the fact that the arithmetic sum

$$\mathcal{S}_0\left(\frac{-1}{1 - e^{-i\delta}}\right) = \sum_{n=1}^{\infty} d(n)\, \mathrm{e}\left(-\frac{n}{1 - e^{-i\delta}}\right)$$

decays exponentially fast for $\delta \to 0^+$, while the Fourier series $h(\delta)$ is very rapidly convergent.

The situation is somewhat analogous to the situation of the partition function $p(n)$. Hardy and Ramanujan found an asymptotic series for $p(n)$ and subsequently Rademacher gave a series which was both asymptotic and convergent. In both the partition case and our case, the exact formula allows for the computation of the sought quantity to *any* desired degree of precision, whereas an asymptotic series has limits to its precision. Of course, an extra feature of $p(n)$, that is not present in our situation, is that since $p(n)$ is an integer it is known exactly once it is known to a precision of 0.5. However, our formula does have the extra surprising feature that the time required to calculate our desired mean square is basically independent of δ, apart from the intrinsic difficulty of the extra work required just to write down a high precision number δ.

11.2 Proof of Theorem 11.1.1

In this section we prove the exact formula for the second moment of the Riemann zeta function.

Proof of Theorem 11.1.1. Firstly, observe that

$$L_1(\delta) = -ie^{-\frac{i\delta}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} \zeta(s)\zeta(1-s)e^{i\delta s}\, \mathrm{d}s.$$

The functional equation for $\zeta(s)$,

$$\zeta(1-s) = \chi(1-s)\zeta(s),$$

where

$$\chi(1-s) = (2\pi)^{-s}\Gamma(s)\Big(e^{\frac{\pi is}{2}} + e^{-\frac{\pi is}{2}}\Big),$$

allows us to split $L_1(\delta)$ as

$$\begin{aligned} L_1(\delta) &= -ie^{-\frac{i\delta}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} \chi(1-s)\zeta(s)^2 e^{i\delta s}\, \mathrm{d}s \\ &= -ie^{-\frac{i\delta}{2}}\big(L^+(\delta) + L^-(\delta)\big), \end{aligned}$$

where

$$L^{\pm}(\delta) = \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} (2\pi)^{-s}\Gamma(s)e^{\pm\frac{\pi is}{2}}\zeta(s)^2 e^{i\delta s}\, \mathrm{d}s. \tag{11.2.1}$$

By Stirling's formula $L^+(\delta)$ is analytic for $\Re(\delta) > -\pi$. Moreover, by contour integration,

$$\begin{aligned} L^-(\delta) &= \int_{(2)} (2\pi)^{-s}\Gamma(s)e^{-\frac{\pi is}{2}}\zeta(s)^2 e^{i\delta s}\, \mathrm{d}s - G(\delta) \\ &= J(\delta) - G(\delta), \end{aligned}$$

say, where

$$\begin{aligned} G(\delta) := &\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}} (2\pi)^{-s}\Gamma(s)e^{-\frac{\pi is}{2}}\zeta(s)^2 e^{i\delta s}\, \mathrm{d}s + \\ &+ 2\pi i \operatorname{Res}_{s=1}\Big((2\pi)^{-s}\Gamma(s)e^{-\frac{\pi is}{2}}\zeta(s)^2 e^{i\delta s}\Big) \end{aligned} \tag{11.2.2}$$

is analytic for $\Re(\delta) < \pi$. Now, expanding $\zeta(s)^2$ into its Dirichlet series, for $\Re(\delta) > 0$ we have

$$\begin{aligned} J(\delta) &= \sum_{n=1}^{\infty} d(n) \int_{2-i\infty}^{2+i\infty} \Gamma(s)(2\pi i n e^{-i\delta})^{-s} \,\mathrm{d}s \\ &= 2\pi i \mathcal{S}_0\bigl(-e^{-i\delta}\bigr) = 2\pi i \mathcal{S}_0\bigl(1-e^{-i\delta}\bigr). \end{aligned} \tag{11.2.3}$$

By Theorem 7.1.1, we can write this as

$$J(\delta) = \frac{\log 2\pi\delta - \gamma}{1-e^{-i\delta}} - \pi g_0\bigl(1-e^{-i\delta}\bigr) + \frac{2\pi i}{1-e^{-i\delta}}\mathcal{S}_0\left(\frac{-1}{1-e^{-i\delta}}\right) + ie^{i\delta}\omega(\delta),$$

where

$$\omega(\delta) = -\frac{\log\left(\frac{1-e^{-i\delta}}{\delta}\right) - \frac{\pi i}{2}}{2\sin\left(\frac{\delta}{2}\right)}$$

is holomorphic in $|\Re(\delta)| < \pi$. Summing up, we have

$$\begin{aligned} L_1(\delta) =& \frac{\gamma - \log 2\pi\delta}{2\sin\frac{\delta}{2}} + \frac{\pi i}{\sin\frac{\delta}{2}}\mathcal{S}_0\left(\frac{-1}{1-e^{-i\delta}}\right) + i\pi e^{-\frac{i\delta}{2}} g_0\bigl(1-e^{-i\delta}\bigr) \\ &+ \omega(\delta) - ie^{-\frac{i\delta}{2}}\bigl(L^+(\delta) - G(\delta)\bigr). \end{aligned} \tag{11.2.4}$$

The theorem then follows after writing

$$h(\delta) := i\pi e^{-i\frac{\delta}{2}} g_0(1-e^{-i\delta})$$

and applying Theorem 7.1.1 and 7.1.2. $\square$

Chapter 12

An exact formula for the twisted second moment of the Riemann zeta function

12.1 Introduction

As discussed in Part A, Chapter 2, the study of moments plays a major role in the theory of L-functions. In particular, for many applications it is important to have a good understanding of the mean-value of an L-function multiplied by a Dirichlet polynomial. In this chapter we consider the case of the second moment of the Riemann zeta function times a Dirichlet polynomial $A(s) := \sum_{m\leq Y} a_m m^{-s}$,

$$I_A(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t. \tag{12.1.1}$$

If $|a_m| \ll m^{\varepsilon}$ for all $\varepsilon > 0$, it is known that $I_A(T)$ is equal to

$$I_A(T) = T\sum_{h,k\leq Y}\frac{a_h\overline{a_k}}{hk}(h,k)\left(\log\frac{T(h,k)^2}{2\pi hk}+2\gamma-1\right)+o(T), \tag{12.1.2}$$

provided that $Y \leq T^{\frac{1}{2}-\delta}$ for some small $\delta > 0$, when the range of uniformity in Y can be extended to $Y \leq T^{\frac{4}{7}-\delta}$, for any $\delta > 0$, if one assumes Hooley's

Conjecture R^* (see [BCH-B]). It is conjectured that (12.1.2) holds in the wider range $Y \leq T^{1-\delta}$, whereas it is known that this conjecture doesn't hold true in full generality for Dirichlet polynomial of length greater than $T^{1+\delta}$. For certain choices of coefficients a_n one can extend unconditionally the range of validity of (12.1.2). For example, Balasubramanian, Conrey and Heath-Brown [BCH-B] could extend the uniformity up to $Y^{\frac{9}{17}}$ when taking coefficients a_m of the form $\mu(m)F(m)$ for smooth functions F that obey some decay conditions.

Results of this type have direct applications to the computation of lower bounds for the percentage of zeros of ζ that lie on the critical line. For example, in [Con] the proof that this percentage is greater than 40% is based on the extension of the uniformity up to $Y \leq T^{\frac{4}{7}-\delta}$ for particular coefficients in the case when ζ is slightly off the $\frac{1}{2}$ line. Another important case is that of coefficients of the form $a_m = m^{i\alpha}$ for $0 \leq \alpha \leq \frac{1}{\delta}$. In fact, the extension of (12.1.1) to a range $Y \leq T^{\frac{1}{2}+\delta}$ (or even an upper bound of roughly the right order) for this choice of coefficients could be easily used to deduce a sub-convexity bound for ζ, moreover reaching an uniformity up to $Y \leq T^{1-\varepsilon}$ would imply the Lindelöf hypothesis.

In the same way as for the smooth second moment of ζ (see Section 11.1), essentially the same considerations made for $I_A(T)$ hold when considering the smoothed integral

$$J_A(\delta) := \int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 e^{-\delta t}\,\mathrm{d}t. \tag{12.1.3}$$

In this chapter we will follow the same methods used in Chapter 11 to provide an exact formula for

$$J\left(\delta, \frac{h}{k}\right) := \int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{h}{k}\right)^{it} e^{-\delta t}\,\mathrm{d}t;$$

summing over h and k will then give an exact formula for $J_A(\delta)$. To do this we express $J\big(\delta, \frac{h}{k}\big)$ in terms of the arithmetic sum

$$\mathcal{S}\left(z, \frac{h}{k}\right) := \sum_{n\geq 1} \mathrm{e}\left(n\frac{h}{k}\right) d(n)\,\mathrm{e}(nz),$$

for some z with $\Im(z) > 0$, and then apply Lemma 10.2.1. Before we state the theorem, we define

$$\begin{aligned} g\left(z, \frac{h}{k}\right) &:= g_{0,0}\left(z, \frac{h}{k}\right), \\ \mathcal{S}\left(z, \frac{h}{k}\right) &:= \mathcal{S}\left(z, 0, \frac{h}{k}\right), \end{aligned}$$

with $g_{0,0}$ and $\mathcal{S}\big(z, 0, \frac{h}{k}\big)$ as in (10.1.4) and (10.2.1). In particular, $g\big(z, \frac{h}{k}\big)$ is an holomorphic function of z in $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Theorem 12.1.1. *Let* $(h, k) = 1$ *and* $0 < \Re(\delta) < \pi$. *Let*

$$\mu(z, k) := \frac{1}{4} - \frac{1}{4z} + \frac{\log(2\pi k z) - \gamma}{2\pi i z},$$

for $z \in \mathbb{C}'$. *Then*

$$\begin{aligned} J\left(\delta, \frac{h}{k}\right) = 2\pi e^{-\frac{i\delta}{2}} \sqrt{\frac{h}{k}} \Bigg(& \mu\big(h\big(1 - e^{-i\delta}\big), k\big) \\ & + \frac{1}{h(1 - e^{-i\delta})} \mathcal{S}\left(-\frac{1}{kh(1 - e^{-i\delta})}, \frac{\overline{h}}{k}\right) \\ & - \frac{i}{2} \mathrm{c}_0\left(\frac{h}{k}\right) + \frac{i}{2} g\left(h\big(1 - e^{-i\delta}\big), -\frac{h}{k}\right)\Bigg) + f\left(\delta - i \log \frac{k}{h}\right), \end{aligned} \tag{12.1.4}$$

where $\mathrm{c}_0\big(\frac{h}{k}\big)$ *is the cotangent sum defined in* (6.0.3), $\overline{h}$ *is the inverse of* h *modulo* k *and* $f(\delta)$ *is analytic in* $|\Re(\delta)| < \pi$ *and bounded in* $|\Re(\delta)| < \pi - \varepsilon$ *for all* $\varepsilon > 0$.

We remark that the arithmetic sum $\mathcal{S}\left(-\frac{1}{kh\left(1-e^{-i\delta}\right)}, \frac{\overline{h}}{k}\right)$ is rapidly convergent and exponentially small as $\delta \to 0^+$ (for fixed h and k).

Multiplying (12.1.4) by $\frac{a_h \overline{a_k}}{\sqrt{hk}}$ and summing over h and k one obtains an exact formula for $J_A(\delta)$. From this it is easy to obtain an asymptotic formula in the case where the length of the Dirichlet polynomial is less than $\big(\frac{1}{\delta}\big)^{\frac{1}{2}-\varepsilon}$, as we shall see in the next section.

Corollary 12.1.2. *Let* $\Re(\delta) > 0$. *Moreover, let* $a_h \ll h^\varepsilon$ *and* $1 \leq Y \ll |\delta|^{-\frac{1}{2}+\varepsilon}$ *for all* $\varepsilon > 0$. *Then*

$$J_A(\delta) = \frac{1}{\delta} \sum_{h,k \leq Y} \frac{a_h \overline{a_k}}{hk} (h,k) \left(\log \frac{(h,k)^2}{2\pi hk\delta} - \gamma \right) + O\left(Y^{2+\varepsilon} + \frac{1}{|\delta|^\varepsilon} \right)$$

as δ *goes to* 0.

We conclude the chapter by giving a generalization of Theorem 7.1.2, showing, in particular, that the coefficients of the Taylor series of $g\big(z, \frac{h}{k}\big)$ when $z = h$ are "very small".

Theorem 12.1.3. *For* $|y| < 1$ *let*

$$g\left(h + hy, -\frac{h}{k} \right) = \sum_{m=0}^{\infty} g_m(h,k)(-y)^m, \tag{12.1.5}$$

be the Taylor series of $g\big(z, -\frac{h}{k}\big)$ *at* $z = h$. *Then,*

$$\begin{aligned} g_m(h,k) = \frac{2^{\frac{7}{4}} k^{\frac{1}{4}}}{\pi^{\frac{3}{4}} h^{\frac{3}{4}}} \frac{e^{-2\sqrt{\frac{\pi m}{hk}}}}{m^{\frac{1}{4}}} \cos\left(\frac{\pi}{hk} - 2\sqrt{\frac{\pi m}{hk}} - \frac{\pi}{8} + 2\pi \frac{\overline{h}}{k} \right) + \\ + O\left(\frac{k^{\frac{3}{4}}}{n^{\frac{3}{4}} h^{\frac{1}{4}}} e^{-2\sqrt{\frac{\pi n}{hk}}} \right), \end{aligned} \tag{12.1.6}$$

uniformly in $h, k, m \geq 1$. *In particular, the Taylor series* (12.1.5) *is convergent on* $|z| = 1$.

We remark this theorem allows us to write the function $g\big(h\big(1 - e^{i\delta}\big), -\frac{h}{k}\big)$ appearing in Theorem 12.1.1 as a Fourier series,

$$g\left(h\big(1 - e^{i\delta}\big), -\frac{h}{k} \right) = \sum_{n \geq 0} (-1)^n g_n(h,k) e^{-in\delta},$$

if $\Im(\delta) \leq 0$. Notice also that (12.1.6) implies that this function is C^∞ for all $\delta \in \mathbb{R}$.

12.2 The twisted second moment

In this section we prove Theorem 12.1.1 and Corollary 12.1.2.

Proof of Theorem 12.1.1. Following the proof of Theorem 11.1.1, it is easy to see that, for $0 < \Re(z) < \pi$, one has

$$\begin{aligned} L_1(z) :=& \int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 e^{-zt}\,\mathrm{d}t \\ =& 2\pi e^{-\frac{iz}{2}} \sum_{n\geq 1} d(n)\,\mathrm{e}\big(-ne^{-iz}\big) + f(z), \end{aligned} \tag{12.2.1}$$

where

$$f(z) := -ie^{-iz}\big(L^+(z) - G(z)\big),$$

and $L^+(z)$ and $G(z)$ are as in (11.2.1) and (11.2.2). From the definition of G and L^+ it is clear that $f(z)$ is bounded and analytic in $|\Re(z)| < \pi - \varepsilon$ for all $\varepsilon > 0$.

Thus, applying (12.2.1) to $z = \delta + i\log\frac{h}{k}$ and writing $x = h\big(1 - e^{-i\delta}\big)$, we have

$$\begin{aligned} \int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{h}{k}\right)^{it} e^{-\delta t}\,\mathrm{d}t &= 2\pi e^{-\frac{iz}{2}} \sum_{n=1}^{\infty} d(n)\,\mathrm{e}\left(-n\frac{h}{k}\right)\mathrm{e}\left(n\frac{x}{k}\right) + f(z) \\ &= 2\pi e^{-\frac{iz}{2}} \mathcal{S}\left(\frac{x}{k}, -\frac{h}{k}\right) + f(z). \end{aligned}$$

Applying Lemma 10.2.1 (taking $a = M = 0$), we have that this is

$$2\pi e^{-\frac{iz}{2}}\left(\frac{1}{x}\mathcal{S}\left(-\frac{1}{kx}, \frac{\overline{h}}{k}\right) + r_{0,0}\left(x, -\frac{h}{k}\right) + \frac{i}{2}g\left(x, -\frac{h}{k}\right)\right) + f(z)$$

and the theorem then follows, since by (9.2.2) one has

$$D\left(0, 0, -\frac{h}{k}\right) = \frac{1}{4} - \frac{i}{2}\mathrm{c}_0\left(\frac{h}{k}\right).$$

□

Proof of Corollary 12.1.2. Firstly, we write

$$h^* := \frac{h}{(h,k)}, \qquad k^* := \frac{k}{(h,k)}.$$

and we assume δ is a bounded real number with positive real part.

Multiplying (12.1.4) by $\frac{a_h\overline{a_k}}{\sqrt{hk}}$ and summing over h and k, we get

$$\begin{aligned} J_A(\delta) &= \sum_{h,k\leq Y}\frac{a_h\overline{a_k}}{\sqrt{hk}}\int_0^\infty\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left(\frac{k^*}{h^*}\right)^{it}e^{-\delta t}\,\mathrm{d}t \\ &= A+B+C+S+T+U, \end{aligned} \tag{12.2.2}$$

where

$$\begin{aligned} A &:= -\sum_{h,k\leq Y}a_h\overline{a_k}\frac{\log\left(-2\pi i h^*k^*\left(1-e^{-i\delta}\right)\right)-\gamma}{2h^*k\sin\frac{\delta}{2}} \\ B &:= 2\pi e^{\frac{i\delta}{2}}\sum_{h,k\leq Y}\frac{a_h\overline{a_k}}{h^*k(1-e^{-i\delta})}\sum_{n\geq 1}d(n)\,\mathrm{e}\left(n\overline{h^*}/k^*\right)\exp\left(-\frac{2\pi i n}{k^*h^*(1-e^{-i\delta})}\right), \\ C &:= \pi i e^{\frac{i\delta}{2}}\sum_{h,k\leq Y}\frac{a_h\overline{a_k}}{k}g\left(h^*\left(1-e^{-i\delta}\right),-\frac{h}{k}\right), \\ S &:= \pi i e^{\frac{i\delta}{2}}\sum_{h,k\leq Y}\frac{a_h\overline{a_k}}{k}\,\mathrm{c}_0\left(-\frac{h^*}{k^*}\right) \end{aligned}$$

and

$$\begin{aligned} T &:= \frac{\pi e^{-\frac{i\delta}{2}}}{2}\sum_{h,k\leq Y}\frac{a_h\overline{a_k}}{k}, \\ U &:= \sum_{h,k\leq Y}\frac{a_h\overline{a_k}}{\sqrt{hk}}f\left(\delta-i\log\frac{h}{k}\right). \end{aligned}$$

Clearly,

$$\begin{aligned} T &\ll \sum_{h,k\leq Y}\frac{(hk)^\varepsilon}{k}\ll Y^{1+2\varepsilon}, \\ U &\ll \sum_{h,k\leq Y}\frac{(hk)^\varepsilon}{\sqrt{hk}}\ll Y^{1+2\varepsilon}, \end{aligned}$$

since $|f(z)|\ll 1$ for $\Re(z)$ small enough. Moreover, one can easily check that $\mathrm{c}_0\left(\frac{h}{k}\right)\ll k\log k$ and thus we have that

$$S\ll Y^{2+\varepsilon}.$$

(Notice that if $a_h\in\mathbb{R}$ then the error term coming from S becomes negligible if one considers the real part in (12.2.2). We remark also that one could show that $\Im(S)$ is always $O(Y^{1+\varepsilon})$ by using the reciprocity formula (9.1.3) for c_0 and a trivial bound for ψ_0).

Now, applying the functional equation (9.2.1) for D in the definition (10.1.4) of $g_{0,0}$, for $|\arg z| < \pi - \varepsilon$ (and after moving the line of integration to $\Re(s) = -\varepsilon$) we get that

$$g_{0,0}\left(z, -\frac{h}{k}\right) \ll (k^*)^{\frac{1}{2}} \int_{(-\varepsilon)} e^{\left(\frac{3}{2}\pi - \varepsilon\right)\frac{|s|}{2}} \frac{|\Gamma(1-s)|}{|\sin \pi s|} |k^* z|^{\Re(s)} \, |\mathrm{d}s| \ll (k^*)^{1+\varepsilon} |z|^{\varepsilon},$$

since

$$D\left(\frac{3}{2} + it, \pm\frac{\overline{h^*}}{k^*}\right) \ll 1.$$

Thus,

$$C \ll \sum_{h,k \ll Y} (hk)^{2\varepsilon} |\delta|^{\varepsilon} \ll Y^{4+2\varepsilon} |\delta|^{\varepsilon}.$$

Moreover,

$$B \ll \sum_{h,k \leq Y} \frac{1}{|\delta|(h^*k^*)^{1-\varepsilon}} \left| \sum_{n \geq 1} d(n) \, e^{-\frac{2\pi n}{h^* k^* \delta}} \right| \ll \sum_{h,k \leq Y} (hk)^{2\varepsilon} |\delta|^{\varepsilon} \ll Y^{2+4\varepsilon} |\delta|^{\varepsilon}.$$

Finally, for $Y \ll \delta^{-1+\varepsilon}$, one has

$$A = -\sum_{h,k \leq Y} a_h \overline{a_k} \frac{\log(2\pi h^* k^* \delta) - \gamma}{h^* k \delta} + O((Y/|\delta|)^{\varepsilon})$$

and the Corollary then follows. □

12.3 The Taylor coefficients

In this section we give a proof of Theorem 12.1.3, following the same method used to prove Theorem 7.1.2.

Proof of Theorem 12.1.3. For $m \geq 1$ we have

$$\begin{aligned} g^{(m)}\left(z, \frac{h}{k}\right) &= \frac{(-1)^m}{\pi i} \int_{\left(-\frac{1}{2}\right)} \frac{\Gamma(s)(2\pi/k)^{-s} z^{-s-m}}{\sin \pi s} \frac{\Gamma(s+m)}{\Gamma(s)} \times \\ &\quad \times \left(e^{-\frac{\pi i s}{2}} D\left(s, 0, \frac{h}{k}\right) + e^{\frac{\pi i s}{2}} D\left(s, 0, -\frac{h}{k}\right)\right) \mathrm{d}s \qquad (12.3.1) \\ &= \frac{(-1)^m}{\pi i} \int_{\left(-\frac{1}{2}\right)} \frac{z^{-s-m}}{\sin \pi s} \frac{\Gamma(s+m)}{\Gamma(s)} C\left(s, \frac{h}{k}\right) \mathrm{d}s, \end{aligned}$$

where, for $(h,k)=1$,

$$C\left(s,\frac{h}{k}\right) := \left(\frac{k}{2\pi}\right)^{s}\Gamma(s)\left(e^{-\frac{\pi i s}{2}}D\left(s,0,\frac{h}{k}\right)+e^{\frac{\pi i s}{2}}D\left(s,0,-\frac{h}{k}\right)\right).$$

Notice that $C\left(s,\frac{h}{k}\right)$ satisfies the functional equation

$$C\left(s,\frac{h}{k}\right)=C\left(1-s,\frac{\overline{h}}{k}\right), \tag{12.3.2}$$

since, applying the functional equation (9.2.1) for D (and the reflection formula for the Gamma function), one has

$$\begin{aligned}C\left(s,\frac{h}{k}\right) &= -\frac{1}{k}\left(\frac{k}{2\pi}\right)^{2-s}\Gamma(s)\Gamma(1-s)^2\Bigg((e^{-\frac{3\pi i s}{2}}-e^{\frac{\pi i s}{2}})D\left(1-s,0,-\frac{\overline{h}}{k}\right)+\\ &\qquad\qquad +(e^{\frac{3\pi i s}{2}}-e^{-\frac{\pi i s}{2}})D\left(1-s,0,\frac{\overline{h}}{k}\right)\Bigg)\\ &=-i\left(\frac{k}{2\pi}\right)^{1-s}\Gamma(1-s)\left(e^{\frac{\pi i s}{2}}D\left(1-s,0,\frac{\overline{h}}{k}\right)-e^{-\frac{\pi i s}{2}}D\left(1-s,0,-\frac{\overline{h}}{k}\right)\right)\\ &=C\left(1-s,\frac{\overline{h}}{k}\right).\end{aligned}$$

Therefore, taking $z=h$, (12.3.1) becomes

$$g^{(m)}\left(h,\frac{h}{k}\right)=\frac{(-1)^m}{\pi^2 i}\int_{\left(-\frac{1}{2}\right)}h^{-s-m}\Gamma(s+m)\Gamma(1-s)C\left(1-s,\frac{\overline{h}}{k}\right)\mathrm{d}s.$$

Expanding the Estermann functions D in the definition of C into Dirichlet series, we have

$$\begin{aligned}g^{(m)}\left(h,\frac{h}{k}\right) &= \frac{(-1)^m k}{\pi^2 i h^m}\sum_{\ell\ge 1}\frac{d(\ell)}{\ell}\frac{1}{2\pi i}\int_{\left(-\frac{1}{2}\right)}\Gamma(s+m)\Gamma(1-s)^2\left(\frac{2\pi\ell}{hk}\right)^{s}\times\\ &\qquad\times\left(i^s\,\mathrm{e}\left(\frac{\ell\overline{h}}{k}\right)-(-i)^s\,\mathrm{e}\left(-\frac{\ell\overline{h}}{k}\right)\right)\mathrm{d}s\\ &=\frac{(-1)^m k}{\pi^2 i h^m}\sum_{\ell\ge 1}\frac{d(\ell)}{\ell}\left(\mathrm{e}\left(\frac{\ell\overline{h}}{k}\right)\Upsilon_m^{+}\left(\frac{\ell}{hk}\right)-\mathrm{e}\left(-\frac{\ell\overline{h}}{k}\right)\Upsilon_m^{-}\left(\frac{\ell}{hk}\right)\right)\end{aligned}$$

where $\Upsilon_m^{\pm}(x)=\Upsilon_{m,0}^{\pm}(x)$ is defined in (7.3.3). As in the proof of Theorem 7.1.2 the contribution coming from $\ell\ge 2$ is negligible, whereas for $\ell=1$ we use (7.3.4) which, if $m\ge 1$, gives

$$\Upsilon_m^{\pm}\left(\frac{1}{hk}\right)=\pm 2i\frac{2^{\frac{1}{4}}\pi^{\frac{7}{4}}}{(hk)^{\frac{3}{4}}}\,e^{\pm\pi i\left(\frac{1}{hk}-\frac{1}{8}\right)-2(1\pm i)\sqrt{\pi m/hk}}m^{m+\frac{1}{4}}e^{-m}\left(1+O\left(\sqrt{\frac{hk}{m}}\right)\right),$$

uniformly in $h, k \geq 1$. Therefore, by Stirling's formula,

$$\begin{aligned}
\frac{g^{(m)}\big(h, \frac{h}{k}\big)}{m!} &= (-1)^m \frac{2^{\frac{3}{4}} k^{\frac{1}{4}} h^{-m-\frac{3}{4}}}{\pi^{\frac{3}{4}}} \frac{e^{-2\sqrt{\frac{\pi m}{hk}}}}{m^{\frac{1}{4}}} \times \\
&\quad \times \left(2\Re\Big(e^{\pi i\left(\frac{1}{hk} - \frac{1}{8}\right) - 2i\sqrt{\frac{\pi m}{hk}} + 2\pi i \frac{\overline{h}}{k}} \Big) + O\left(\sqrt{\frac{hk}{m}}\right)\right) \\
&= (-1)^m \frac{2^{\frac{7}{4}} k^{\frac{1}{4}} h^{-m-\frac{3}{4}}}{\pi^{\frac{3}{4}}} \frac{e^{-2\sqrt{\frac{\pi m}{hk}}}}{m^{\frac{1}{4}}} \times \\
&\quad \times \left(\cos\left(\frac{\pi}{hk} - 2\sqrt{\frac{\pi m}{hk}} - \frac{\pi}{8} + 2\pi \frac{\overline{h}}{k} \right) + O\left(\sqrt{\frac{hk}{m}}\right)\right)
\end{aligned}$$

uniformly in $h, k \geq 1$ and the Theorem follows. $\square$

Part C

On the Nyman-Beurling-Báez-Duarte criterion

Chapter 13

An optimal choice of Dirichlet polynomials for the Nyman-Beurling-Báez-Duarte criterion

The work presented in this chapter is joint with J.B Conrey and D.W. Farmer and was first published in [BCF].

13.1 Introduction

In Part B, Chapter 9 we have introduced the Báez-Duarte [Bae] reformulation of the the Nyman-Beurling criterion for the Riemann hypothesis. We recall here that the Báez-Duarte criterion asserts that the Riemann hypothesis is true if and only if

$$\lim_{N\to\infty} d_N^2 = 0,$$

where

$$d_N^2 := \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} |1 - \zeta A_N(1/2 + it)|^2 \frac{dt}{\frac{1}{4} + t^2}$$

and the infimum is over all Dirichlet polynomials $A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ of length at most N.

A question that arises naturally from this criterion is to determine what the rate of convergence of d_n to zero is, assuming the Riemann hypothesis. Balazard and de Roton showed that, if the Riemann hypothesis is true, then

$$d_N^2 \ll \frac{(\log\log N)^{\frac{5}{2}+\varepsilon}}{\sqrt{\log N}},$$

for all $\epsilon > 0$. On the other hand Báez-Duarte, Balazard, Landreau and Saias [BBLS00, BBLS05] showed (unconditionally) that d_N^2 can not decay faster than a constant times $\frac{1}{\log N}$. More precisely, they showed that

$$\liminf_{N\to\infty} d_N^2 \log N \geq \sum_{\Re(\rho)=1/2} \frac{1}{|\rho|^2},$$

where the sum is restricted to distinct zeros of the Riemann zeta function on the critical line. The constant was later improved by Burnol [Bur] who showed

$$\liminf_{N\to\infty} d_N^2 \log N \geq \sum_{\Re(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2},$$

where $m(\rho)$ denotes the multiplicity of ρ. This lower bound is believed to be optimal and one expects that

$$d_N^2 \sim \frac{1}{\log N} \sum_{\Re(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2}. \qquad (13.1.1)$$

Notice that under the Riemann hypothesis, one has

$$\sum_{\Re(\rho)=1/2} \frac{m(\rho)}{|\rho|^2} = 2 + \gamma - \log 4\pi$$

and in particular, if all the non-trivial zeros of $\zeta(s)$ are simple, then (13.1.1) can be rewritten as

$$d_N^2 \sim \frac{2 + \gamma - \log 4\pi}{\log N}.$$

In this chapter we shall prove (13.1.1) under the Riemann Hypothesis and assuming a mild condition on the growth of the mean value of $\frac{1}{|\zeta'(\rho)|^2}$ over the non-trivial zeros $|\rho| \leq T$ of $\zeta(s)$. This will be achieved by using the Dirichlet polynomial

$$V_N(s) := \sum_{n=1}^{N} \left(1 - \frac{\log n}{\log N}\right) \frac{\mu(n)}{n^s}.$$

Theorem 13.1.1. *If the Riemann hypothesis is true and if*

$$\sum_{|\Im(\rho)| \leq T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{\frac{3}{2}-\delta} \tag{13.1.2}$$

for some fixed $\delta > 0$, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |1 - \zeta V_N(1/2 + it)|^2 \frac{dt}{\frac{1}{4} + t^2} \sim \frac{2 + \gamma - \log 4\pi}{\log N}.$$

The condition (13.1.2) implicitly assumes that the zeros of the Riemann zeta function are all simple. Moreover, this upper bound is "mild" in the sense that a conjecture, due to Gonek and recovered by a different heuristic method of Hughes, Keating, and O'Connell [HKO], predicts that

$$\sum_{|\rho| \leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{6}{\pi^3} T.$$

We remark that Theorem 13.1.1 is in contrast to what one might have expected after viewing the graphs of Landreau and Richards [LR] which at first sight suggest that V_N is not optimal.

This behaviour of the Riemann zeta function resembles that of polynomials. In fact, Grenander and Rosenblatt [GR] (see also Theorem 2.1 in [Bur]) showed that for a polynomial $P(z)$ one has that the zeros of P are all located outside or on the unit circle if and only if $\lim_{N \to \infty} \delta_N = 0$, where

$$\delta_N^2 = \frac{1}{2\pi} \inf_{Q_N} \int_0^{2\pi} \left|1 - P(z)\, Q_N(z)\right|^2 \mathrm{d}\theta,$$

where $z = e^{i\theta}$ and the infimum is over polynomials Q_N of degree at most N. Moreover, if this happens, then

$$\lim_{N \to \infty} N\delta_N^2 = \sum_{|\rho|=1} m(\rho)^2,$$

where the sum is restricted to the distinct zeros ρ of $P(z)$ lying on the unit circle and $m(\rho)$ is again the multiplicity of ρ.

This analogy seems to apply also to the choice of optimal polynomials.

Theorem 13.1.2. *Let $P(z)$ be a polynomial whose zeros are all simple and lie outside or on the unit circle. Let*

$$W_N(z) := \sum_{n=0}^{N} \Big(1 - \frac{n}{N}\Big) a_n z^n, \tag{13.1.3}$$

where

$$\frac{1}{P(z)} = \sum_{n \geq 0} a_n z^n$$

is the Taylor expansion in $x = 0$ of the inverse of $P(z)$ (i.e. it is the formal power series inverse of $P(z)$). Then

$$\frac{1}{2\pi} \int_0^{2\pi} |1 - P(z) W_N(z)|^2 \, \mathrm{d}\theta \sim \frac{1}{N} \sum_{|\rho|=1} m(\rho)^2,$$

where $z = e^{i\theta}$.

We remark that the proofs of Theorem 13.1.1 and 13.1.2 are very similar, the main difference being that the Riemann zeta function has infinitely many zeros. This generates some issues concerning the convergence of certain sums of $\frac{1}{\zeta'(\rho)}$, which force us to assume condition (13.1.2).

13.2 Polynomials

Lemma 13.2.1. *Let $P(s)$ be a polynomial with $P(0) \neq 0$. We have*

$$W_N(s) = \frac{1}{P(s)} \bigg(1 + \frac{s}{N} \frac{P'}{P}(s)\bigg) - \frac{s}{N} Y_N(s),$$

where $W_N(s)$ is defined in (13.1.3)*,*

$$Y_N(s) := \sum_{\rho}^{*} \operatorname*{Res}_{z=\rho} \frac{s^N}{P(z)(z-s)^2 z^N}$$

and the sum is over distinct zeros ρ of $P(z)$.

Proof. Since $P(0) \neq 0$, we can take an $\varepsilon > 0$ such that all the zeros of $P(z)$ lie outside of the circle $|z| = \varepsilon$. Now, observe that we can assume $0 < |s| < \varepsilon$, since the result will then extend to all $\mathbb{C}$ by analytic continuation. Denoting by $\mathcal{C}_y$ the circle of radius $y > 0$ (oriented in the positive direction), by the residue theorem we have that

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \frac{\mathrm{d}z}{z^{n+1}},$$

therefore

$$W_N(s) = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \sum_{n=0}^{N} \Big(1 - \frac{n}{N}\Big) \Big(\frac{s}{z}\Big)^n \frac{\mathrm{d}z}{z}.$$

Now,

$$\sum_{n=0}^{N} \Big(1 - \frac{n}{N}\Big) z^n = -\frac{1}{N} \frac{z - z^{N+1}}{(1-z)^2} + \frac{1}{1-z}$$

and thus

$$W_N(s) = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \Big(-\frac{1}{N} \frac{s z^N - s^{N+1}}{(z-s)^2 z^N} + \frac{1}{z-s} \Big) \mathrm{d}z.$$

Now, by the residue theorem

$$\frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \Big(-\frac{1}{N} \frac{s}{(z-s)^2} + \frac{1}{z-s} \Big) \mathrm{d}z = \frac{1}{P(s)} \Big(1 + \frac{s}{N} \frac{P'(s)}{P(s)} \Big),$$

whereas, moving the line of integration to $\mathcal{C}_y$ and letting y tends to infinity, one has that

$$\frac{1}{2\pi i N} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \frac{s^{N+1}}{(z-s)^2 z^N} \mathrm{d}z = -\frac{s}{N} Y_N(s)$$

and the Lemma follows. □

Proof of Theorem 13.1.2. Let $\delta > 1$ be such that $P(s)$ does not have any zero on $1 < |s| \leq \delta$. We have

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \big|1 - P(e^{i\theta}) W_N(e^{i\theta})\big|^2 \mathrm{d}\theta = \\ &\quad = \frac{1}{2\pi i} \int_{\mathcal{C}_1} (1 - P(s) W_N(s)) \Big(1 - \overline{P}\Big(\frac{1}{s}\Big) \overline{W}_N\Big(\frac{1}{s}\Big) \Big) \frac{\mathrm{d}s}{s} \\ &\quad = \frac{1}{2\pi i} \int_{\mathcal{C}_\delta} (1 - P(s) W_N(s)) \Big(1 - \overline{P}\Big(\frac{1}{s}\Big) \overline{W}_N\Big(\frac{1}{s}\Big) \Big) \frac{\mathrm{d}s}{s}. \end{aligned}$$

Therefore, by Lemma 13.2.1, this is

$$\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\bigg(\frac{P'}{P}(s)-P(s)Y_N(s)\bigg)\bigg(\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)-\overline{P}\Big(\frac{1}{s}\Big)\overline{Y}_N\Big(\frac{1}{s}\Big)\bigg)\,\frac{\mathrm{d}s}{s}.$$

Now, for $|s|=\delta$ one has

$$Y_N(s)\overline{Y}_N\Big(\frac{1}{s}\Big)=O(1),$$

therefore

$$\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\bigg(\frac{P'}{P}(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)+P(s)Y_N(s)\overline{P}\Big(\frac{1}{s}\Big)\overline{Y}_N\Big(\frac{1}{s}\Big)\bigg)\,\frac{\mathrm{d}s}{s}=O\Big(\frac{1}{N^2}\Big).$$

Moreover for $s\in\mathcal{C}_\delta$ one has that $\overline{Y}_N\big(\frac{1}{s}\big)=O\big(\delta^{-N}\big)$, thus

$$-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta}\bigg(\frac{P'}{P}(s)\overline{P}\Big(\frac{1}{s}\Big)\overline{Y}_N\Big(\frac{1}{s}\Big)\bigg)\,\frac{\mathrm{d}s}{s}=O\big(\delta^{-N}/N^2\big).$$

Finally, by the residue theorem,

$$\begin{aligned}
-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_\delta} & P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\,\frac{\mathrm{d}s}{s}= \\
&=-\frac{1}{N^2}\sum_{|\rho|=1}\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\frac{1}{s}+ \\
&\quad-\frac{1}{2\pi i N^2}\int_{\mathcal{C}_{\frac{1}{\delta}}}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\,\frac{\mathrm{d}s}{s} \\
&=-\frac{1}{N^2}\sum_{|\rho|=1}\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)+O\big(\delta^{-N}/N^2\big).
\end{aligned}$$

The theorem then follows by observing that

$$\operatorname*{Res}_{s=\rho}P(s)Y_N(s)\frac{\overline{P}'}{\overline{P}}\Big(\frac{1}{s}\Big)\frac{1}{s}=-N+O(1).$$

□

13.3 The Riemann zeta-function

We start with the following lemma, which is the analogue of Lemma 13.2.1. We remark that this lemma is unconditional.

Lemma 13.3.1. *If* $0 < \Re(s) < 1$, *then*

$$V_N(s) = \frac{1}{\zeta(s)}\left(1 - \frac{1}{\log N}\frac{\zeta'}{\zeta}(s)\right) + \frac{1}{\log N}\sum_{\rho}{}^{*} R_N(\rho, s) + \frac{1}{\log N}F_s(1/N),$$

where the sum is over distinct non-trivial zeros ρ *of* $\zeta(s)$ *with*

$$R_N(\rho, s) = \operatorname*{Res}_{z=\rho} \frac{N^{z-s}}{\zeta(z)(z-s)^2},$$

and where

$$F_s(z) = \pi z^s \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n+1} z^{2n}}{(2n)!\zeta(2n+1)(2n+s)^2}$$

is an entire function of z.

Proof. We have

$$V_N(s) = \frac{1}{\log N}\frac{1}{2\pi i}\int_{(2)} \frac{N^w}{\zeta(s+w)}\frac{dw}{w^2}.$$

Now we move the path of integration to $\Re(w) = -\Re(s) - 2M - 1$ for a large positive integer M. The residue at $w = \rho - s$ is $R_N(\rho, s)/\log N$. The residue at $s + w = -2n$ is

$$\frac{N^{-2n-s}}{\zeta'(-2n)(2n+s)^2 \log N}$$

and the integral on the new path is $\ll N^{-2M-1}$. Letting $M \to \infty$ and using

$$\zeta'(-2n) = \frac{(-1)^n \pi (2n)! \zeta(2n+1)}{(2\pi)^{2n+1}}$$

we obtain the result. □

Lemma 13.3.2. *Let* $\varepsilon > 0$. *Assume the Riemann hypothesis and that all the zeros of* $\zeta(s)$ *are simple. Then, if condition* (13.1.2) *holds, for* $\Re(s) = \frac{1}{2} \pm \varepsilon$ *one has*

$$\sum_{\rho} R_N(\rho, s) \ll N^{\mp\varepsilon} |s|^{\frac{3}{4} - \frac{\delta}{2} + \varepsilon}. \tag{13.3.1}$$

Proof. Firstly observe that, by the Cauchy-Schwartz inequality, (13.1.2) implies

$$\sum_{|\rho|\le T} \frac{1}{|\zeta'(\rho)|} \ll \sqrt{N(T) \sum_{|\rho|\le T} \frac{1}{|\zeta'(\rho)|^2}} \ll T^{\frac{5}{4}-\frac{\delta}{2}}\sqrt{\log T},$$

since

$$N(T) := \frac{1}{2}\sum_{|\rho|\le T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

Therefore, by partial summation, we have that the series

$$\sum_{\rho} \frac{1}{|\zeta'(\rho)||\rho|^{\alpha}}$$

is convergent for any $\alpha > \frac{5}{4} - \frac{\delta}{2}$. Now, for a simple zero ρ, we have

$$R_N(\rho, s) = \frac{N^{\rho-s}}{\zeta'(\rho)(\rho-s)^2}.$$

Therefore

$$\begin{aligned} N^{\pm\varepsilon}\sum_{\rho} R_N(\rho,s) &\ll \sum_{|\rho-s|<\frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho-s|^2} + \sum_{|\rho-s|\ge\frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho-s|^2} \\ &\ll \sum_{|\rho-s|<\frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho-s|^2} + \sum_{|\rho-s|\ge\frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho|^2} \qquad (13.3.2) \\ &\ll \sum_{|\rho-s|<\frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho-s|^2} + 1. \end{aligned}$$

Now, by the Cauchy-Schwartz inequality,

$$\sum_{|\rho-s|<\frac{|\rho|}{2}} \frac{1}{|\zeta'(\rho)||\rho-s|^2} \ll \left(\sum_{|\rho|<2|s|} \frac{1}{|\zeta'(\rho)|^2}\right)^{\frac{1}{2}} \left(\sum_{|\rho|<2|s|} \frac{1}{|\rho-s|^4}\right)^{\frac{1}{2}} \ll |s|^{\frac{3}{4}-\frac{\delta}{2}+\varepsilon},$$

since, by partial summation,

$$\sum_{|\rho|<2|s|} \frac{1}{|\rho-s|^4} \ll \log(|s|+2).$$

This completes the proof of the lemma. □

Proof of Theorem 13.1.1. We have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}&|1-\zeta V_N(1/2+it)|^2\frac{dt}{1/4+t^2}\\ &=\frac{1}{2\pi i}\int_{\left(\frac{1}{2}\right)}(1-\zeta V_N(s))(1-\zeta V_N(1-s))\,\frac{ds}{s(1-s)}\\ &=\frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\varepsilon\right)}(1-\zeta V_N(s))(1-\zeta V_N(1-s))\,\frac{ds}{s(1-s)}.\end{aligned}$$

By Lemma 13.3.1, this is

$$\begin{aligned}\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\varepsilon\right)}&\left(\frac{\zeta'}{\zeta^2}(s)-\sum_\rho R_N(\rho,s)-F_s\left(\frac{1}{N}\right)\right)\times\\ &\times\left(\frac{\zeta'}{\zeta^2}(1-s)-\sum_\rho R_N(\rho,1-s)-F_{1-s}\left(\frac{1}{N}\right)\right)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds.\end{aligned}\tag{13.3.3}$$

Now, we have

$$\begin{aligned}\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\varepsilon\right)}&\sum_{\rho_1,\rho_2}R_N(\rho_1,s)R_N(\rho_2,1-s)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds\\ &\ll\frac{1}{\log^2 N}\int_{\left(\frac{1}{2}-\varepsilon\right)}\sum_{|\rho-s|<\frac{|\rho|}{2}}\frac{1}{|\zeta'(\rho)||\rho-s|^2}\,\frac{|ds|}{|s|^{\frac{5}{4}+\frac{\delta}{2}-5\varepsilon}}+O\left(\frac{1}{\log^2 N}\right),\end{aligned}$$

where we used (13.3.1), (13.3.2) and the bound $\zeta\left(\frac{1}{2}\pm\varepsilon\pm it\right)\ll|t|^{2\varepsilon}$ (which is a consequence of the Lindelöf hypothesis). Reversing the order of summation and integration, we have that this is bounded by

$$\begin{aligned}\frac{1}{\log^2 N}\sum_\rho\frac{1}{|\zeta'(\rho)|}&\int_{\left(\frac{1}{2}-\varepsilon\right)+i\left(\Im(\rho)-\frac{|\rho|}{2}\right)}^{\left(\frac{1}{2}-\varepsilon\right)+i\left(\Im(\rho)+\frac{|\rho|}{2}\right)}\frac{|ds|}{|\rho-s|^2|s|^{\frac{5}{4}+\frac{\delta}{2}-5\varepsilon}}+O\left(\frac{1}{\log^2 N}\right)\\ &\ll\frac{1}{\log^2 N}\sum_\rho\frac{1}{|\zeta'(\rho)||\rho|^{\frac{5}{4}+\frac{\delta}{2}-5\varepsilon}}\ll\frac{1}{\log^2 N},\end{aligned}$$

if $\varepsilon<\frac{\delta}{10}$.

Now, by Lemma 13.3.2 and the trivial estimate $F_s(z)=O\left(N^{-\frac{5}{2}}\right)$, all the other terms in (13.3.3) are trivially $O\left(\frac{1}{\log^2 N}\right)$ apart from

$$-\frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\varepsilon\right)}\frac{\zeta'}{\zeta}(1-s)\sum_\rho R_N(\rho,s)\frac{\zeta(s)}{s(1-s)}\,ds.\tag{13.3.4}$$

The integrand has a double pole at every zero ρ of residue

$$\mathrm{Res}_{s=\rho}\left(\frac{\zeta'}{\zeta}(1-s)\sum_{\rho} R_N(\rho,s)\frac{\zeta(s)}{s(1-s)}\right) = \frac{\log N - \frac{1}{2}\frac{\zeta''(\rho)}{\zeta'(\rho)} + \frac{\chi'}{\chi}(\rho) + \frac{1-2\rho}{|\rho|^2}}{|\rho|^2}$$
$$= \frac{\log N}{|\rho|^2} + O\left(\frac{1}{|\rho|^{2-\varepsilon}|\zeta'(\rho)|} + \frac{1}{|\rho|^2}\right),$$

where we used the bound $\zeta''\left(\frac{1}{2}+it\right) \ll |t|^{\varepsilon}$, which follows from the Lindelöf hypothesis and Cauchy's estimate for the derivatives of a holomorphic function. It follows that moving the line of integration in (13.3.4) to $\Re(s) = \frac{1}{2}+\varepsilon$ we get that the integral is equal to

$$\frac{1}{\log N}\sum_{\rho}\frac{1}{|\rho|^2} + O\left(\frac{1}{\log^2 N}\right),$$

and Theorem 13.1.1 then follows. $\square$

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